

# Supersymmetry, chiral symmetry and the generalized BRS transformation in lattice formulations of 4D $\mathcal{N} = 1$ SYM

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## Abstract

In the context of the lattice regularization of the four-dimensional  $\mathcal{N} = 1$  supersymmetric Yang–Mills theory (4D  $\mathcal{N} = 1$  SYM), we formulate a generalized BRS transformation that treats the gauge, supersymmetry (SUSY), translation and axial  $U(1)$  ( $U(1)_A$ ) transformations in a unified way. A resultant Slavnov–Taylor identity or the Zinn–Justin equation gives rise to a strong constraint on the quantum continuum limit of symmetry breaking terms with the lattice regularization. By analyzing the implications of the constraint on operator-mixing coefficients in the SUSY and the  $U(1)_A$  Ward–Takahashi (WT) identities, we prove to all orders of perturbation theory in the continuum limit that, (i) the chiral symmetric limit implies the supersymmetric limit and, (ii) a three-fermion operator that might potentially give rise to an exotic breaking of the SUSY WT identity does not emerge. In previous literature, only a naive or incomplete treatment on these points can be found. Our results provide a solid theoretical basis for lattice formulations of the 4D  $\mathcal{N} = 1$  SYM.

*Keywords:* Supersymmetry, Chiral symmetry, Lattice gauge theory

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## 1. Introduction

Ever since the feasibility of non-perturbative studies of 4D  $\mathcal{N} = 1$  SYM—the simplest but still quite non-trivial 4D supersymmetric gauge theory—

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based on the lattice regularization was pointed out [1],<sup>1</sup> there has been considerable research based on the proposed scenario of the SUSY restoration. Classified by the method for simulating the gaugino or gluino, a fermionic superpartner of the gauge boson, Refs. [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22] use the Wilson fermion [23] (see Ref. [24] for a very readable review), Refs. [25, 26, 27, 28] employ the domain-wall fermion [29, 30] and, most recently, Ref. [31] uses the overlap fermion [32, 33]. Some related works are Refs. [34, 35, 36, 37, 38].<sup>2</sup>

The central issue in the scenario [1] is the restoration of SUSY and the axial  $U(1)$  symmetry,  $U(1)_A$ , in the quantum continuum limit; the lattice regularization generally breaks these symmetries that define 4D  $\mathcal{N} = 1$  SYM. The basic physical picture of Ref. [1] is very simple (see also Ref. [2]): In terms of the effective field theory, a unique relevant operator that breaks SUSY is the gluino mass term. If one can have a  $U(1)_A$  symmetry (either by the fine-tuning of the bare gluino mass [40, 41, 42] or by use of the Ginsparg–Wilson relation [43]; see below), the  $U(1)_A$  would forbid the gluino mass term and one would end up with a supersymmetric theory in the continuum limit. In this way, SUSY might emerge as an accidental symmetry accompanied by the chiral symmetry. The situation is a little bit more complicated because the  $U(1)_A$  suffers from the axial anomaly. Still, one could use a remaining  $\mathbb{Z}_{2N_c} \subset U(1)_A$  symmetry<sup>3</sup> to forbid the gluino mass term. Thus one expects that, in the continuum limit, the restoration of SUSY and that of the chiral symmetry occur simultaneously.

The best characterization of a symmetry property of a quantum field theory is given by the WT identity. Although the basic physical picture of the above scenario is simple, to show the validity of the scenario in terms of WT identities is not so simple.<sup>4</sup> For example, the analysis of Ref. [1] considers a lattice “identity”, such as (Eq. (14) of Ref. [1] with  $\Omega = A_\rho(y)\bar{\lambda}(z)$ , in our

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<sup>1</sup>See also Ref. [2] for an earlier consideration.

<sup>2</sup>Ref. [39] is discussing a possible implication of the lattice BRS invariance (in the non-perturbative level) in supersymmetric gauge theories.

<sup>3</sup>Throughout this paper, we assume that the gauge group is  $SU(N_c)$ .

<sup>4</sup>For a lattice formulation [44, 45] of the two-dimensional  $\mathcal{N} = (2, 2)$  SYM, the restoration of the SUSY WT identity without fine-tuning can be argued perturbatively [46] and has been confirmed non-perturbatively [47].

notation)

$$\begin{aligned} & \langle \partial_\mu^* S_\mu(x) A_\nu^a(y) \bar{\psi}^b(z) \rangle \\ & \stackrel{?}{=} \langle [M\chi(x) + X_S(x)] A_\nu^a(y) \bar{\psi}^b(z) \rangle - \left\langle \frac{1}{a^4} \frac{\partial}{\partial \bar{\xi}(x)} \delta_\xi [A_\nu^a(y) \bar{\psi}^b(z)] \right\rangle, \quad (1.1) \end{aligned}$$

where  $S_\mu(x)$  is the supercurrent on the lattice,  $X_S(x)$  is an explicit SUSY breaking term in the lattice action,  $A_\nu^a(y)$  is the gauge potential and  $\bar{\psi}^b(z)$  is the gluino field.  $\delta_\xi$  in the last term is the SUSY transformation on the lattice. This relation, however, does *not* hold even in the tree level approximation of the perturbation theory.<sup>5</sup> The point is that the inserted operator  $A_\nu^a(y) \bar{\psi}^b(z)$  is neither gauge invariant nor covariant thus the contribution of the gauge-fixing term (and the ghost action for higher orders) cannot be neglected. Moreover, as found in Ref. [37] through an explicit one-loop calculation,  $X_S(x)$  mixes even with gauge non-invariant operators, a situation that was not presumed in Ref. [1]. In these aspects, the WT identity associated with SUSY is much more complicated than that for the chiral symmetry [41, 42] as noted sometime ago [15]. It is certainly unlikely that those gauge non-invariant elements affect a physical question. Nevertheless, it is not quite obvious whether or not these complications modify the conclusion of Ref. [1] that both the  $U(1)_A$  WT identity (with the axial anomaly) and the SUSY WT identity are restored by a single fine-tuning of the bare gluino mass. Since the axial  $U(1)$  current and the supercurrent belong to a single SUSY multiplet (at least in the classical theory) [48], it is quite conceivable that one can relate the  $U(1)_A$  WT identity and the SUSY WT identity by considering an algebra involving both SUSY and the  $R$ -symmetry,  $U(1)_A$ . It is not a priori clear how to carry out this program, however, especially with the lattice regularization that generally breaks both symmetries explicitly.

The original motivation of the present work was to obtain a transparent understanding on the above issue. In the course of the analysis, however, we encountered another difficulty that has not been noted in previous literature. That is that, generally, one might have a mixing of  $X_S(x)$  with a gauge invariant three-fermion operator. Since this three-fermion operator has the same mass-dimension as  $X_S(x)$  and same transformation properties as  $X_S(x)$  under lattice symmetries, a dimensional counting and a simple symmetry argument alone cannot exclude the possibility of such a mixing. But then,

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<sup>5</sup>I would like to thank Yusuke Taniguchi for noticing this observation.

one would have an exotic breaking of the SUSY WT identity even in the continuum limit (and even with the fine-tuning of the gluino mass elucidated above). See Eq. (4.20). According to the supposed scenario of the SUSY restoration, we should not have such an exotic breaking.

Considering recent increases in computing power that would enable a true realization of 4D  $\mathcal{N} = 1$  SYM and a fundamental role that the SUSY WT identity should play there, it must be important to clarify the above issues; this is the objective of the present paper. We will find that the adoption of the generalized BRS transformation [49, 50, 51] in the lattice framework, that treats the gauge, SUSY, translation and  $U(1)_A$  transformations in a unified way, does the required job.

This paper is organized as follows. In Sec. 2, we summarize basic assumptions we make on the lattice formulation. They are physically natural and very mild; actually all the lattice formulations of 4D  $\mathcal{N} = 1$  SYM considered so far are covered by our analysis. Sections. 3 and 4 are mostly a review of well-known facts concerning the operator mixing in the  $U(1)_A$  WT identity [41, 42] and the SUSY WT identity [1, 37, 15]; however, we tried not to make any ad hoc assumption on the possible operator-mixing structure. Eqs. (4.22) and (4.23) summarize our main assertions in this paper. After these rather long preparations, in Sec. 5, we introduce the generalized BRS transformation on lattice variables. There, some complications inherent in the present supersymmetric system in the Wess–Zumino (WZ) gauge are explained. After identifying  $O(a)$  breaking terms attributed to the lattice regularization, we obtain a Wess–Zumino (WZ)-like consistency condition [52] by using the algebraic property of the generalized BRS transformation. In Sec. 6, finally, combining the consistency condition and the general structure of the operator mixing explored in Secs. 3 and 4, we prove the main assertions, Eqs. (4.22) and (4.23). Sec. 7 concludes the paper. Appendices contain some useful information required in the main text.

## 2. Basic lattice framework

Our arguments below are quite general and rather independent of a particular lattice formulation we adopt.<sup>6</sup> Our lattice action consists of the sum of a gauge boson (gluon) action  $S_{\text{gluon}}$ , the kinetic term of the gluino  $S_{\text{gluino}}$

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<sup>6</sup>We basically follow the notation of Ref. [37] except the points that  $x, y, z, \dots$  denote lattice sites, vector indices  $\mu, \nu, \dots$ , run over 0, 1, 2, 3 and the lattice spacing  $a$  is explicitly

and the mass term of the gluino,  $S_{\text{mass}}^{(0)}$ . A minimal requirement for the lattice actions is that they have the correct *classical* continuum limit such that

$$S_{\text{gluon}} \xrightarrow{a \rightarrow 0} \check{S}_{\text{gluon}} \equiv \int d^4x \frac{1}{2} \text{tr} [F_{\mu\nu}(x) F_{\mu\nu}(x)], \quad (2.4)$$

where  $F_{\mu\nu}(x)$  is the field strength  $F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + ig[A_\mu(x), A_\nu(x)]$ ,

$$S_{\text{gluino}} \equiv a^4 \sum_x \text{tr} [\bar{\psi}(x) D \psi(x)] \xrightarrow{a \rightarrow 0} \check{S}_{\text{gluino}} \equiv \int d^4x \text{tr} [\bar{\psi}(x) \not{D} \psi(x)], \quad (2.5)$$

where  $D$  is a lattice Dirac operator and  $\not{D} \equiv \gamma_\mu D_\mu$  and  $D_\mu \equiv \partial_\mu + ig[A_\mu(x), \cdot]$  is the covariant derivative with respect to the adjoint representation, and

$$S_{\text{mass}}^{(0)} \equiv a^4 \sum_x M \text{tr} [\bar{\psi}(x) \psi(x)] \xrightarrow{a \rightarrow 0} \check{S}_{\text{mass}}^{(0)} \equiv \int d^4x M \text{tr} [\bar{\psi}(x) \psi(x)], \quad (2.6)$$

where  $M$  is the bare gluino mass. The gluino field  $\psi(x)$  on the Euclidean lattice (and in the Euclidean continuum theory as well) is subject to the constraint

$$\bar{\psi}(x) = \psi^T(x)(-C^{-1}), \quad (2.7)$$

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written. The gauge potentials  $A_\mu(x)$  are defined from the link variables by

$$U(x, \mu) = e^{iagA_\mu(x)}, \quad (2.1)$$

where  $g$  is the bare gauge coupling constant. All gamma matrices are hermitian and obey  $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ ;  $\gamma_5 \equiv -\gamma_0\gamma_1\gamma_2\gamma_3$  and  $\sigma_{\mu\nu} \equiv [\gamma_\mu, \gamma_\nu]/2$ . The charge conjugation matrix  $C$  satisfies,  $C^{-1}\gamma_\mu C = -\gamma_\mu^T$ ,  $C^{-1}\sigma_{\mu\nu} C = -\sigma_{\mu\nu}^T$ ,  $C^{-1}\gamma_5 C = \gamma_5^T$  and  $C^T = -C$ .  $\epsilon_{\mu\nu\rho\sigma}$  denotes the totally antisymmetric tensor and  $\epsilon_{0123} = -1$ . The generators of the gauge group  $SU(N_c)$ ,  $T^a$ , are normalized as  $\text{tr}(T^a T^b) = (1/2)\delta^{ab}$ .  $\Phi^a(x)$  denotes the gauge component of a generic field  $\Phi(x)$ ,  $\Phi(x) \equiv \Phi^a(x)T^a$ . Throughout this article, the symbol  $\text{tr}$  denotes the trace over gauge indices. The forward and backward difference operators respectively are defined by

$$\partial_\mu f(x) \equiv \frac{1}{a} [f(x + a\hat{\mu}) - f(x)], \quad \partial_\mu^* f(x) \equiv \frac{1}{a} [f(x) - f(x - a\hat{\mu})]. \quad (2.2)$$

The symbol of the functional derivative with respect to a lattice variable implies

$$\frac{\delta}{\delta \Phi^a(x)} \equiv \frac{1}{a^4} \frac{\partial}{\partial \Phi^a(x)}, \quad (2.3)$$

for  $\Phi^a(x)$ , for example.

to express the degrees of freedom of the Majorana fermion in the Minkowski space. Consequently, the lattice Dirac operator should satisfy  $(C^{-1}D)^T = -C^{-1}D$ . We further assume that the lattice actions are invariant under the hypercubic transformations, the parity transformation  $\mathcal{P}$ ,

$$\begin{aligned} U_0(x_0, \vec{x}) &\xrightarrow{\mathcal{P}} U_0(x_0, -\vec{x}), & U_k(x_0, \vec{x}) &\xrightarrow{\mathcal{P}} U_k^\dagger(x_0, -\vec{x} - a\hat{k}), \\ \psi(x_0, \vec{x}) &\xrightarrow{\mathcal{P}} i\gamma_0\psi(x_0, -\vec{x}), & \bar{\psi}(x_0, \vec{x}) &\xrightarrow{\mathcal{P}} -i\bar{\psi}(x_0, -\vec{x})\gamma_0, \end{aligned} \quad (2.8)$$

and the time reversal transformation  $\mathcal{T}$ ,

$$\begin{aligned} U_0(x_0, \vec{x}) &\xrightarrow{\mathcal{T}} U_0^\dagger(-x_0 - a, \vec{x}), & U_k(x_0, \vec{x}) &\xrightarrow{\mathcal{T}} U_k(-x_0, \vec{x}), \\ \psi(x_0, \vec{x}) &\xrightarrow{\mathcal{T}} i\gamma_0\gamma_5\psi(-x_0, \vec{x}), & \bar{\psi}(x_0, \vec{x}) &\xrightarrow{\mathcal{T}} -i\bar{\psi}(-x_0, \vec{x})\gamma_5\gamma_0. \end{aligned} \quad (2.9)$$

The simplest lattice formulation which fulfills the above requirements is the sum of the Wilson plaquette action,

$$S_{\text{gluon}} = \sum_x \sum_{\mu, \nu} \left( -\frac{1}{g^2} \right) \text{Re tr} [U_\mu(x)U_\nu(x + a\hat{\mu})U_\mu^\dagger(x + a\hat{\nu})U_\nu^\dagger(x)], \quad (2.10)$$

and the Wilson fermion action,

$$\begin{aligned} S_{\text{gluino}} &= a^4 \sum_x \text{tr} \left( \bar{\psi}(x) \left\{ \frac{1}{2} \sum_\mu [\gamma_\mu(\nabla_\mu + \nabla_\mu^*) - ra\nabla_\mu^*\nabla_\mu] \right\} \psi(x) \right), \\ S_{\text{mass}}^{(0)} &= a^4 \sum_x M \text{tr} [\bar{\psi}(x)\psi(x)], \end{aligned} \quad (2.11)$$

where lattice covariant differences in the adjoint representation are defined by

$$\begin{aligned} \nabla_\mu \psi(x) &\equiv \frac{1}{a} [U_\mu(x)\psi(x + a\hat{\mu})U_\mu^\dagger(x) - \psi(x)], \\ \nabla_\mu^* \psi(x) &\equiv \frac{1}{a} [\psi(x) - U_\mu^\dagger(x - a\hat{\mu})\psi(x - a\hat{\mu})U_\mu(x - a\hat{\mu})]. \end{aligned} \quad (2.12)$$

In what follows, we often consider correlation functions containing elementary fields that are not gauge invariant. To define such correlation functions, we have to introduce the gauge-fixing term and the Faddeev–Popov (FP) ghost and anti-ghost fields. We thus define

$$S_{\text{GF+FP}}^{(0)} \equiv -s_0 a^4 \sum_x 2 \text{tr} \left\{ \bar{c}(x) \left[ \partial_\mu^* A_\mu(x) + \frac{\alpha}{2} B(x) \right] \right\}, \quad (2.13)$$

where  $\bar{c}(x)$  and  $B(x)$  are the anti-ghost and auxiliary fields, respectively and  $\alpha$  is the gauge parameter. The nilpotent BRS transformation  $s_0$  for lattice variables is defined by (see, for example, Ref. [53]),

$$\begin{aligned} s_0 A_\mu(x) &\equiv [D_\mu c]^L(x), \\ s_0 \psi(x) &\equiv -ig\{c(x), \psi(x)\}, & s_0 \bar{\psi}(x) &= -ig\{c(x), \bar{\psi}(x)\}, \\ s_0 c(x) &\equiv -igc(x)^2, \\ s_0 \bar{c}(x) &\equiv B(x), \\ s_0 B(x) &\equiv 0, \end{aligned} \tag{2.14}$$

where

$$[D_\mu c]^L(x) \equiv \frac{ig\Delta_{A_\mu(x)}}{1 - \exp[-ig\Delta_{A_\mu(x)}]} \partial_\mu c(x) + ig\Delta_{A_\mu(x)} c(x), \tag{2.15}$$

and  $\Delta_X$  is the adjoint action,  $\Delta_X Y \equiv [X, Y]$ . We set

$$\varphi(x_0, \vec{x}) \xrightarrow{\mathcal{P}} \varphi(x_0, -\vec{x}), \quad \varphi(x_0, \vec{x}) \xrightarrow{\mathcal{T}} \varphi(-x_0, \vec{x}), \tag{2.16}$$

for all  $\varphi(x) \equiv c(x)$ ,  $\bar{c}(x)$  and  $B(x)$ . Then one can verify that  $[D_\mu c]^L(x)$  in Eq. (2.15) behaves in an identical way as  $A_\mu(x)$  under  $\mathcal{P}$  and  $\mathcal{T}$  and, consequently,  $s_0$  preserves transformation properties under  $\mathcal{P}$  and  $\mathcal{T}$ . This shows that (since the combination  $\partial_\mu^* A_\mu(x)$  behaves as Eq. (2.16))  $S_{\text{GF+FP}}^{(0)}$  (2.13) is invariant under  $\mathcal{P}$  and  $\mathcal{T}$ . Our total lattice action is thus given by

$$S^{(0)} \equiv S_{\text{gluon}} + S_{\text{gluino}} + S_{\text{mass}}^{(0)} + S_{\text{GF+FP}}^{(0)}, \tag{2.17}$$

and the corresponding classical continuum limit is

$$\check{S}^{(0)} \equiv \check{S}_{\text{gluon}} + \check{S}_{\text{gluino}} + \check{S}_{\text{mass}}^{(0)} + \check{S}_{\text{GF+FP}}^{(0)}, \tag{2.18}$$

where  $\check{S}_{\text{GF+FP}}^{(0)}$  is the classical continuum limit of  $S_{\text{GF+FP}}^{(0)}$  (2.13).

### 3. $U(1)_A$ WT identity on the lattice

#### 3.1. Derivation of the $U(1)_A$ WT identity

Let us begin our discussion with a WT identity associated with the  $U(1)_A$  transformation. We define a localized version of the  $U(1)_A$  transformation on lattice variables by

$$\delta_\theta \psi(x) = i\theta(x) \gamma_5 \psi(x), \quad \delta_\theta \bar{\psi}(x) = i\theta(x) \bar{\psi}(x) \gamma_5, \tag{3.1}$$

and  $\delta_\theta = 0$  on other variables. Then, from a variation of the lattice action (note that  $S_{\text{GF+FP}}^{(0)}$  (2.13) does not contain the gluino field),

$$\delta_\theta S^{(0)} \equiv a^4 \sum_x i\theta(x) \left[ -\partial_\mu^* j_{5\mu}(x) + 2MP(x) + X_A(x) \right], \quad (3.2)$$

we obtain the divergence of the axial-vector current  $j_{5\mu}(x)$  whose classical continuum limit is given by<sup>7</sup>

$$j_{5\mu}(x) \xrightarrow{a \rightarrow 0} \check{j}_{5\mu}(x) \equiv \text{tr} [\bar{\psi}(x) \gamma_\mu \gamma_5 \psi(x)], \quad (3.3)$$

the pseudo-scalar density  $P(x)$ , which arises from the variation of the gluino mass term  $S_{\text{mass}}^{(0)}$  (2.11),

$$P(x) \equiv \text{tr} [\bar{\psi}(x) \gamma_5 \psi(x)], \quad (3.4)$$

and  $X_A(x)$ , a  $U(1)_A$  symmetry breaking term associated with the lattice regularization (e.g., the Wilson term, for the case of Eq. (2.11)). In Eq. (3.2), the separation between  $-\partial_\mu^* j_{5\mu}(x)$  and  $X_A(x)$  is not unique and there remains  $O(a)$  ambiguity in the definition of  $X_A(x)$  even with requirement (3.3). We partially fix this ambiguity by requiring that the breaking  $X_A(x)$  is a gauge-invariant local combination of lattice fields that behaves in an identical way as  $P(x)$  (3.4) under lattice discrete transformations. In particular, we assume that

$$X_A(x_0, \vec{x}) \xrightarrow{\mathcal{P}} -X_A(x_0, -\vec{x}), \quad X_A(x_0, \vec{x}) \xrightarrow{\mathcal{T}} -X_A(-x_0, \vec{x}). \quad (3.5)$$

Then by considering the variation of the lattice action (3.2) in the functional integral, we find the identity

$$\begin{aligned} & \langle \partial_\mu^* j_{5\mu}(x) \mathcal{O}(y, z, \dots) \rangle \\ &= \langle [2MP(x) + X_A(x)] \mathcal{O}(y, z, \dots) \rangle + i \left\langle \frac{1}{a^4} \frac{\partial}{\partial \theta(x)} \delta_\theta \mathcal{O}(y, z, \dots) \right\rangle, \end{aligned} \quad (3.6)$$

where  $\mathcal{O}(y, z, \dots)$  is any multi-local operator. This is an identity that exactly holds on the lattice.

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<sup>7</sup>A possible form for  $j_{5\mu}(x)$  is  $j_{5\mu}(x) = \text{tr} [\bar{\psi}(x) \gamma_\mu \gamma_5 U_\mu(x) \psi(x + a\hat{\mu}) U_\mu^\dagger(x)]$  [37].



### 3.2. $X_A(x)$ in the continuum limit

We next consider, on the basis of the perturbation theory, how the  $U(1)_A$  breaking term  $X_A(x)$  in Eq. (3.6) behaves in the quantum continuum limit. Since  $X_A(x)$  is a dimension 4 operator that is proportional to the lattice spacing  $a$  (because it arises from lattice artifacts), we set

$$X_A(x) \equiv a\mathcal{O}_5(x), \quad (3.7)$$

where  $\mathcal{O}_5(x)$  is a dimension 5 operator. Then, from dimension counting and the covariance under lattice symmetries (3.5), subtractions which are required to define a renormalized composite operator  $\mathcal{O}_5^R(x)$  in the continuum limit are given by [41] [see also Ref. [42]],

$$\begin{aligned} \mathcal{O}_5^R(x) = \mathcal{Z}_5 \Big\{ & \mathcal{O}_5(x) + \frac{1}{a}(\mathcal{Z}_A - 1)\partial_\mu \check{j}_{5\mu}(x) + \frac{1}{a}\mathcal{Z}_{F\tilde{F}}\epsilon_{\mu\nu\rho\sigma} \text{tr} [F_{\mu\nu}(x)F_{\rho\sigma}(x)] \\ & + \frac{1}{a^2}\mathcal{Z}_P\check{P}(x) + \frac{1}{a}(\text{dim. 4 BRS non-invariant operators}) \Big\} \\ & + \sum_j \mathcal{Z}_5^{(j)}\mathcal{O}_5^{(j)R}(x), \end{aligned} \quad (3.8)$$

where the explicit form of power-subtraction operators is written only for BRS invariant ones.<sup>8</sup> As indicated in Eq. (3.8), generally, one might have a mixing with BRS non-invariant operators depending on the insertion operator  $\mathcal{O}(y, z, \dots)$  in Eq. (3.6). The last line of Eq. (3.8) represents a possible mixing with other (renormalized) dimension 5 operators. We have used the continuum theory language in the above expression ( $\check{P}(x)$  is the classical continuum limit of  $P(x)$  (3.4)), because the present considerations are meaningful only in the continuum limit.

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<sup>8</sup>One sees that there is no dimension 4 (or less) BRS invariant operator that contains  $c(x)$ ,  $\bar{c}(x)$  or  $B(x)$  and complies with property (3.5).

Eqs. (3.7) and (3.8) show that

$$X_A(x) = (1 - \mathcal{Z}_A) \partial_\mu \check{J}_{5\mu}(x) \quad (3.9a)$$

$$- \mathcal{Z}_{F\tilde{F}} \epsilon_{\mu\nu\rho\sigma} \text{tr} [F_{\mu\nu}(x) F_{\rho\sigma}(x)] \quad (3.9b)$$

$$- \frac{1}{a} \mathcal{Z}_P \check{P}(x) \quad (3.9c)$$

$$+ (\text{dim. 4 BRS non-invariant operators}) \quad (3.9d)$$

$$+ a \mathcal{Z}_5^{-1} \left[ \mathcal{O}_5^R(x) - \sum_j \mathcal{Z}_5^{(j)} \mathcal{O}_5^{(j)R}(x) \right], \quad (3.9e)$$

and the lattice identity (3.6) then reads

$$\mathcal{Z}_A \langle \partial_\mu \check{J}_{5\mu}(x) \mathcal{O}(y, z, \dots) \rangle \quad (3.10a)$$

$$= - \mathcal{Z}_{F\tilde{F}} \langle \epsilon_{\mu\nu\rho\sigma} \text{tr} [F_{\mu\nu}(x) F_{\rho\sigma}(x)] \mathcal{O}(y, z, \dots) \rangle \quad (3.10b)$$

$$+ 2 \left( M - \frac{1}{2a} \mathcal{Z}_P \right) \langle \check{P}(x) \mathcal{O}(y, z, \dots) \rangle \quad (3.10c)$$

$$+ \langle (\text{dim. 4 BRS non-invariant operators}) \mathcal{O}(y, z, \dots) \rangle \quad (3.10d)$$

$$+ \left\langle a \mathcal{Z}_5^{-1} \left[ \mathcal{O}_5^R(x) - \sum_j \mathcal{Z}_5^{(j)} \mathcal{O}_5^{(j)R}(x) \right] \mathcal{O}(y, z, \dots) \right\rangle \quad (3.10e)$$

$$+ i \left\langle \frac{1}{a^4} \frac{\partial}{\partial \theta(x)} \delta_\theta \mathcal{O}(y, z, \dots) \right\rangle. \quad (3.10f)$$

Now, for simplicity, we assume *temporarily* that the inserted operator  $\mathcal{O}(y, z, \dots)$  is gauge (or BRS) invariant and renormalizable without subtracting power-divergent terms. That is,

$$\mathcal{O}^R(y, z, \dots) = \mathcal{Z}_\mathcal{O} \mathcal{O}(y, z, \dots) + \sum_j \mathcal{Z}^{(j)} \mathcal{O}^{(j)R}(y, z, \dots), \quad (3.11)$$

where all renormalization constants are at most logarithmically divergent. We further assume that all the points  $x, y, z, \dots$ , in Eq. (3.10) differ from each other. In this situation, the contact term (3.10f) vanishes. The contribution of the dimension 5 operators (3.10e) also vanishes in the continuum limit because of the overall factor of  $a$  (when all the points  $x, y, z, \dots$ , differ, no  $O(1/a)$  ultraviolet divergence arises that can cancel the overall factor  $a$ ). There is no mixing of  $X_A(x)$  with BRS non-invariant operators because the

inserted operator  $\mathcal{O}(y, z, \dots)$  is BRS invariant. Therefore, in this assumed situation, we have

$$\begin{aligned} \mathcal{Z}_A \langle \partial_\mu \check{j}_{5\mu}(x) \mathcal{O}(y, z, \dots) \rangle &= -\mathcal{Z}_{F\tilde{F}} \langle \epsilon_{\mu\nu\rho\sigma} \text{tr} [F_{\mu\nu}(x) F_{\rho\sigma}(x)] \mathcal{O}(y, z, \dots) \rangle \\ &\quad + 2 \left( M - \frac{1}{2a} \mathcal{Z}_P \right) \langle \check{P}(x) \mathcal{O}(y, z, \dots) \rangle. \end{aligned} \quad (3.12)$$

From this (anomalous) conservation law of the axial  $U(1)$  current, one might infer that for an infinitesimal lattice spacing the  $U(1)_A$  symmetry broken by the lattice regularization is restored by tuning the bare mass parameter  $M$  such that

$$M - \frac{1}{2a} \mathcal{Z}_P \rightarrow 0. \quad (3.13)$$

We call this a chiral symmetric limit [40, 41, 1].

Incidentally, as for the domain-wall fermion [29, 30] and for the overlap fermion [32, 33], if the lattice Dirac operator  $D$  satisfies the GW relation  $\gamma_5 D + D \gamma_5 = a D \gamma_5 D$  [43], it is possible to define  $j_{5\mu}(x)$  in Eq. (3.2) such that  $X_A(x) = a \mathcal{O}_5(x) = a \text{tr} [\bar{\psi}(x) D \gamma_5 D \psi(x)]$ . Suppose that  $M = 0$ . Then, in the correlation function (3.6), contractions of gluino fields in this  $\mathcal{O}_5(x)$  and gluino fields in the inserted operator  $\mathcal{O}(y, z, \dots)$  produce a term being proportional to  $\delta_{x,y} \delta_{x,z}, \dots$ , which does not contribute to the divergence of the operator  $\mathcal{O}_5(x)$ . On the other hand, the contraction of gluino fields within  $\mathcal{O}_5(x)$  produces the combination

$$\mathcal{O}_5(x) \rightarrow \frac{1}{a} \frac{1}{a^4} \frac{1}{2} \text{tr}_D [\hat{\gamma}_5^{aa}(x, x)], \quad (3.14)$$

where  $\text{tr}_D$  is the trace over Dirac indices. The modified chiral matrix in this expression is defined by [54, 55]

$$\hat{\gamma}_5^{ab}(x, y) \equiv \gamma_5 2 \text{tr} [T^a (1 - aD) T^b] \delta_{x,y}. \quad (3.15)$$

From the GW relation, one has the identity  $\sum_{z,c} \hat{\gamma}_5^{ac}(x, z) \hat{\gamma}_5^{cb}(z, y) = \delta^{ab} \delta_{x,y}$  and, from this [54],

$$\sum_x \frac{1}{2} \text{tr}_D [\delta \hat{\gamma}_5^{aa}(x, x)] = 0, \quad (3.16)$$

where  $\delta$  denotes an arbitrary infinitesimal local variation of the gauge field. This relation implies that the vertices that result from the composite operator (3.14) identically vanish when the momentum being conjugate to the

position  $x$  vanishes. In possible subtractions (3.8), this property is shared by operators  $\partial_\mu \check{j}_{5\mu}(x)$  and  $\epsilon_{\mu\nu\rho\sigma} \text{tr}[F_{\mu\nu}(x)F_{\rho\sigma}(x)]$ , but not by  $\check{P}(x)$ . This shows that  $\mathcal{Z}_P = 0$  when  $M = 0$ .<sup>9</sup> That is, when  $D$  satisfies the GW relation, we generally have<sup>10</sup>

$$\mathcal{Z}_P \propto aM. \quad (3.17)$$

This ensures Eq. (3.13) for  $M \rightarrow 0$ .

## 4. SUSY WT identity on the lattice

### 4.1. Derivation of the SUSY WT identity

A lattice WT identity associated with the SUSY transformation can also be derived in a similar way as the above  $U(1)_A$  WT identity. The localized version of the SUSY transformation in the continuum is given by

$$\begin{aligned} \delta_\xi A_\mu(x) &= \bar{\xi}(x) \gamma_\mu \psi(x), \\ \delta_\xi \psi(x) &= -\frac{1}{2} \sigma_{\mu\nu} \xi(x) F_{\mu\nu}(x), \quad \delta_\xi \bar{\psi}(x) = \frac{1}{2} \bar{\xi}(x) \sigma_{\mu\nu} F_{\mu\nu}(x), \end{aligned} \quad (4.1)$$

where the Grassmann-odd parameters  $\xi(x)$  obey  $\bar{\xi}(x) = \xi^T(x)(-C^{-1})$ . Corresponding to this, we may adopt the following localized lattice SUSY transformation,

$$\begin{aligned} \delta_\xi U_\mu(x) &= iag \frac{1}{2} [\bar{\xi}(x) \gamma_\mu \psi(x) U_\mu(x) + \bar{\xi}(x + a\hat{\mu}) \gamma_\mu U_\mu(x) \psi(x + a\hat{\mu})], \\ \delta_\xi U_\mu^\dagger(x) &= -iag \frac{1}{2} [\bar{\xi}(x) \gamma_\mu U_\mu^\dagger(x) \psi(x) + \bar{\xi}(x + a\hat{\mu}) \gamma_\mu \psi(x + a\hat{\mu}) U_\mu^\dagger(x)], \\ \delta_\xi \psi(x) &= -\frac{1}{2} \sigma_{\mu\nu} \xi(x) P_{\mu\nu}(x), \quad \delta_\xi \bar{\psi}(x) = \frac{1}{2} \bar{\xi}(x) \sigma_{\mu\nu} P_{\mu\nu}(x), \end{aligned} \quad (4.2)$$

where  $P_{\mu\nu}(x)$  is the clover plaquette, defined by

$$P_{\mu\nu}(x) = \frac{1}{4} \sum_{i=1}^4 \frac{1}{2ia^2g} [U_{i\mu\nu}(x) - U_{i\mu\nu}^\dagger(x)], \quad (4.3)$$

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<sup>9</sup>On the other hand,  $\mathcal{Z}_{F\bar{F}}$  reproduces the correct axial anomaly [56, 57, 58, 59, 60].

<sup>10</sup>I am grateful to Yoshio Kikukawa for a clarifying discussion on this point.

and

$$\begin{aligned}
U_{1\mu\nu}(x) &\equiv U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu^\dagger(x+a\hat{\nu})U_\nu^\dagger(x), \\
U_{2\mu\nu}(x) &\equiv U_\nu(x)U_\mu^\dagger(x-a\hat{\mu}+a\hat{\nu})U_\nu^\dagger(x-a\hat{\mu})U_\mu(x-a\hat{\mu}), \\
U_{3\mu\nu}(x) &\equiv U_\mu^\dagger(x-a\hat{\mu})U_\nu^\dagger(x-a\hat{\mu}-a\hat{\nu})U_\mu(x-a\hat{\mu}-a\hat{\nu})U_\nu(x-a\hat{\nu}), \\
U_{4\mu\nu}(x) &\equiv U_\nu^\dagger(x-a\hat{\nu})U_\mu(x-a\hat{\nu})U_\nu(x+a\hat{\mu}-a\hat{\nu})U_\mu^\dagger(x).
\end{aligned} \tag{4.4}$$

This definition of  $\delta_\xi$  [37] is advantageous because  $P_{\mu\nu}(x)$  possesses the same transformation properties as the continuum  $F_{\mu\nu}(x)$  under  $\mathcal{P}$  and  $\mathcal{T}$ . That is,

$$P_{0k}(x) \xrightarrow{\mathcal{P}} -P_{0k}(x_0, -\vec{x}), \quad P_{kl}(x) \xrightarrow{\mathcal{P}} P_{kl}(x_0, -\vec{x}), \tag{4.5}$$

and

$$P_{0k}(x) \xrightarrow{\mathcal{T}} -P_{0k}(-x_0, \vec{x}), \quad P_{kl}(x) \xrightarrow{\mathcal{T}} P_{kl}(-x_0, \vec{x}). \tag{4.6}$$

We define that the ghost, anti-ghost and auxiliary fields as singlets under the above (localized) SUSY transformation  $\delta_\xi$ . Then one sees that the BRS transformation  $s_0$  (2.14) and  $\delta_\xi$  anti-commute with each other, and as the consequence, the SUSY variation of the gauge fixing and ghost actions is BRS exact:

$$\delta_\xi S_{\text{GF+FP}}^{(0)} = -s_0 a^4 \sum_x 2 \text{tr} [\bar{c}(x) \partial_\mu^* \delta_\xi A_\mu(x)]. \tag{4.7}$$

The explicit form of  $\delta_\xi A_\mu(x)$ , a transformation of the gauge potential induced by lattice SUSY transformation (4.2), is

$$\begin{aligned}
\delta_\xi A_\mu(x) = \frac{1}{2} \Bigg\{ &\bar{\xi}(x) \gamma_\mu \frac{iag \Delta_{A_\mu(x)}}{\exp[iag \Delta_{A_\mu(x)}] - 1} \psi(x) \\
&+ \bar{\xi}(x+a\hat{\mu}) \gamma_\mu \frac{iag \Delta_{A_\mu(x)}}{1 - \exp[-iag \Delta_{A_\mu(x)}]} \psi(x+a\hat{\mu}) \Bigg\}.
\end{aligned} \tag{4.8}$$

Now, we consider the variation of the lattice action (2.17) under Eq. (4.2). As the general structure of the variation (noting Eq. (4.7)), we have

$$\begin{aligned}
\delta_\xi S^{(0)} \equiv a^4 \sum_x \bar{\xi}(x) \Bigg\{ &-\partial_\mu^* S_\mu(x) + M\chi(x) \\
&-s_0 \sum_y 2 \text{tr} \left[ \bar{c}(y) \partial_\mu^{*y} \frac{\partial}{\partial \xi(x)} \delta_\xi A_\mu(y) \right] + X_S(x) \Bigg\}.
\end{aligned} \tag{4.9}$$

Here,  $\chi(x)$  is a gauge invariant fermionic field, which arises from the variation of the gluino mass term  $S_{\text{mass}}^{(0)}$ ,

$$\chi(x) \equiv \sigma_{\mu\nu} \text{tr} [\psi(x) P_{\mu\nu}(x)] \xrightarrow{a \rightarrow 0} \check{\chi}(x) \equiv \sigma_{\mu\nu} \text{tr} [\psi(x) F_{\mu\nu}(x)]. \quad (4.10)$$

In Eq. (4.9), we defined a lattice supercurrent  $S_\mu(x)$ , whose classical continuum limit is

$$S_\mu(x) \xrightarrow{a \rightarrow 0} \check{S}_\mu(x) \equiv -\sigma_{\rho\sigma} \gamma_\mu \text{tr} [\psi(x) F_{\rho\sigma}(x)]. \quad (4.11)$$

However, in Eq. (4.9), the separation between  $-\partial_\mu^* S_\mu(x)$  and  $X_S(x)$ , a SUSY breaking associated with the lattice regularization, is not unique even under condition (4.11); there remains an  $O(a)$  ambiguity in the definition of  $X_S(x)$ . We thus require that the breaking  $X_S(x)$  is a gauge-invariant local combination of lattice fields which behaves in an identical way as  $\chi(x)$  (4.10) under lattice discrete transformations. In particular, we require that

$$X_S(x_0, \vec{x}) \xrightarrow{\mathcal{P}} i\gamma_0 X_S(x_0, -\vec{x}), \quad X_S(x_0, \vec{x}) \xrightarrow{\mathcal{T}} i\gamma_0 \gamma_5 X_S(-x_0, \vec{x}). \quad (4.12)$$

Then, for any multi-local operator  $\mathcal{O}(y, z, \dots)$ , we have an exact identity on the lattice,

$$\begin{aligned} & \langle \partial_\mu^* S_\mu(x) \mathcal{O}(y, z, \dots) \rangle \\ &= \langle [M\chi(x) + X_S(x)] \mathcal{O}(y, z, \dots) \rangle \\ &= \left\langle s_0 \sum_w 2 \text{tr} \left[ \bar{c}(w) \partial_\mu^{*w} \frac{\partial}{\partial \bar{\xi}(x)} \delta_\xi A_\mu(w) \right] \mathcal{O}(y, z, \dots) \right\rangle \\ &= \left\langle \frac{1}{a^4} \frac{\partial}{\partial \bar{\xi}(x)} \delta_\xi \mathcal{O}(y, z, \dots) \right\rangle. \end{aligned} \quad (4.13)$$

#### 4.2. $X_S(x)$ in the continuum limit

We next investigate, within perturbation theory, how the SUSY breaking term  $X_S(x)$  in Eq. (4.13) behaves in the continuum limit.  $X_S(x)$  is a dimension 9/2 operator that is proportional to the lattice spacing  $a$  (because it results from the lattice regularization). Thus we set

$$X_S(x) = a \mathcal{O}_{11/2}(x), \quad (4.14)$$

where  $\mathcal{O}_{11/2}(x)$  is a dimension 11/2 operator. To define a renormalized finite operator  $\mathcal{O}_{11/2}^R(x)$  in the continuum limit, we generally need the subtraction

of operators of mass-dimension 9/2 or less with power-diverging coefficients. Those operators must, by our assumption, behave identically to  $\chi(x)$  (4.10) under lattice discrete transformations (as Eq. (4.12)). A gauge (or BRS) invariant class of such operators, without containing the ghost, anti-ghost and auxiliary fields, are enumerated in Appendix B of Ref. [15]. Utilizing this result, we have<sup>11</sup>

$$\begin{aligned}
& \mathcal{O}_{11/2}^R(x) \\
&= \mathcal{Z}_{11/2} \left\{ \mathcal{O}_{11/2}(x) + \frac{1}{a}(\mathcal{Z}_S - 1)\partial_\mu \check{S}_\mu(x) + \frac{1}{a}\mathcal{Z}_T\partial_\mu \check{T}_\mu(x) \right. \\
&\quad + \frac{1}{a^2}\mathcal{Z}_\chi \check{\chi}(x) \\
&\quad + \frac{1}{a}\mathcal{Z}_{3F} \text{tr} [\psi(x)\bar{\psi}(x)\psi(x)] \\
&\quad + \frac{1}{a}\mathcal{Z}_{\text{EOM}} \{ \gamma_\mu \text{tr} [\psi(x)D_\nu F_{\mu\nu}(x)] - s_0 \gamma_\mu \text{tr} [\psi(x)\partial_\mu \bar{c}(x)] \} \\
&\quad + \frac{1}{a}(\text{dim. 9/2 BRS invariant operators containing } c, \bar{c} \text{ or } B) \\
&\quad \left. + \frac{1}{a}(\text{dim. 9/2 BRS non-invariant operators}) \right\} \\
&\quad + \sum_j \mathcal{Z}_{11/2}^{(j)} \mathcal{O}_{11/2}^{(j)R}(x), \tag{4.15}
\end{aligned}$$

where

$$\check{T}_\mu(x) \equiv 2\gamma_\nu \text{tr} [\psi(x)F_{\mu\nu}(x)]. \tag{4.16}$$

In the above expression, the last line represents possible mixing with other (renormalized) dimension 11/2 operators. From this and Eq. (4.14), we have

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<sup>11</sup>Here, we have noted the Bianchi identity,  $\epsilon_{\mu\nu\rho\sigma}D_\nu F_{\rho\sigma}(x) = 0$ .

$$X_S(x) = (1 - \mathcal{Z}_S)\partial_\mu \check{S}_\mu(x) \quad (4.17a)$$

$$- \mathcal{Z}_T \partial_\mu \check{T}_\mu(x) \quad (4.17b)$$

$$- \frac{1}{a} \mathcal{Z}_\chi \check{\chi}(x) \quad (4.17c)$$

$$- \mathcal{Z}_{3F} \text{tr} [\psi(x) \bar{\psi}(x) \psi(x)] \quad (4.17d)$$

$$- \mathcal{Z}_{\text{EOM}} \{ \gamma_\mu \text{tr} [\psi(x) D_\nu F_{\mu\nu}(x)] - s_0 \gamma_\mu \text{tr} [\psi(x) \partial_\mu \bar{c}(x)] \} \quad (4.17e)$$

$$+ (\text{dim. } 9/2 \text{ BRS invariant operators containing } c, \bar{c} \text{ or } B) \quad (4.17f)$$

$$+ (\text{dim. } 9/2 \text{ BRS non-invariant operators}) \quad (4.17g)$$

$$+ a \mathcal{Z}_{11/2}^{-1} \left[ \mathcal{O}_{11/2}^R(x) - \sum_j \mathcal{Z}_{11/2}^{(j)} \mathcal{O}_{11/2}^{(j)R}(x) \right], \quad (4.17h)$$

and thus, combined with Eq. (4.13),

$$\left\langle \left[ \mathcal{Z}_S \partial_\mu \check{S}_\mu(x) + \mathcal{Z}_T \partial_\mu \check{T}_\mu(x) \right] \mathcal{O}(y, z, \dots) \right\rangle \quad (4.18a)$$

$$= \left( M - \frac{1}{a} \mathcal{Z}_\chi \right) \langle \check{\chi}(x) \mathcal{O}(y, z, \dots) \rangle \quad (4.18b)$$

$$- \mathcal{Z}_{3F} \langle \text{tr} [\psi(x) \bar{\psi}(x) \psi(x)] \mathcal{O}(y, z, \dots) \rangle \quad (4.18c)$$

$$- \mathcal{Z}_{\text{EOM}} \left\langle \frac{1}{2} \gamma_\mu \psi^a(x) \frac{\delta \check{S}_{\text{tot}}^{(0)}}{\delta A_\mu^a(x)} \mathcal{O}(y, z, \dots) \right\rangle \quad (4.18d)$$

$$+ \langle (\text{dim. } 9/2 \text{ BRS inv. op. containing } c, \bar{c} \text{ or } B) \mathcal{O}(y, z, \dots) \rangle \quad (4.18e)$$

$$+ \langle (\text{dim. } 9/2 \text{ BRS non-inv. op.}) \mathcal{O}(y, z, \dots) \rangle \quad (4.18f)$$

$$+ a \mathcal{Z}_{11/2}^{-1} \left\langle \left[ \mathcal{O}_{11/2}^R(x) - \sum_j \mathcal{Z}_{11/2}^{(j)} \mathcal{O}_{11/2}^{(j)R}(x) \right] \mathcal{O}(y, z, \dots) \right\rangle \quad (4.18g)$$

$$- \left\langle \frac{1}{a^4} \frac{\partial}{\partial \bar{\xi}(x)} \delta_\xi \mathcal{O}(y, z, \dots) \right\rangle, \quad (4.18h)$$

where we have expressed operator (4.17e) in terms of the variations of the total action of the continuum theory,  $\check{S}_{\text{tot}}^{(0)}$  (2.18),<sup>12</sup> and absorbed the  $s_0$ -exact

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<sup>12</sup>Although in Eq. (4.18d) one has another term,  $-ig\gamma_\mu \text{tr}[\psi(x) \bar{\psi}(x) \gamma_\mu \psi(x)]$ , this combination identically vanishes as shown in Appendix A.



term in Eq. (4.13) into Eq. (4.18e). As shown in Eqs. (4.17g) and (4.18f), generally, we might have mixing with BRS non-invariant operators depending on the insertion operator  $\mathcal{O}(y, z, \dots)$ ; this mixing has actually been observed in a one-loop calculation [37].

In Eq. (4.15), and consequently in Eqs. (4.17d) and (4.18c), we have an operator

$$\mathcal{O}_S(x) \equiv \text{tr} [\psi(x) \bar{\psi}(x) \psi(x)] , \quad (4.19)$$

that is cubic in the fermion field. In Refs. [1, 15], on the other hand, one does not encounter such a mixing with a three-fermion operator. In Ref. [1], this operator was not noticed at all, while in Ref. [15], it seems that only the case of the gauge group  $SU(2)$  is considered for which this three-fermion operator identically vanishes  $\mathcal{O}_S(x) \equiv 0$ . The fact is that, as we analyse in detail in Appendix A,  $\mathcal{O}_S$  does not generally vanish, and since  $\mathcal{O}_S(x)$  and  $\chi(x)$  transform in a completely identical manner under lattice symmetries,<sup>13</sup> we cannot exclude  $\mathcal{O}_S(x)$  from the operator renormalization by a simple symmetry argument alone.

For simplicity, as we did in Eq. (3.12), let us *temporarily* assume that the inserted operator  $\mathcal{O}(y, z, \dots)$  in Eq. (4.18) is BRS invariant and renormalizable without subtracting power-divergent terms as Eq. (3.11); we assume also that all the points  $x, y, z, \dots$ , differ. Then, by a similar argument used to obtain Eq. (3.12), we have in the continuum limit

$$\begin{aligned} & \left\langle \left[ \mathcal{Z}_S \partial_\mu \check{S}_\mu(x) + \mathcal{Z}_T \partial_\mu \check{T}_\mu(x) \right] \mathcal{O}(y, z, \dots) \right\rangle \\ &= \left( M - \frac{1}{a} \mathcal{Z}_\chi \right) \langle \check{\chi}(x) \mathcal{O}(y, z, \dots) \rangle \\ & \quad - \mathcal{Z}_{3F} \langle \text{tr} [\psi(x) \bar{\psi}(x) \psi(x)] \mathcal{O}(y, z, \dots) \rangle , \end{aligned} \quad (4.20)$$

where we neglected contribution (4.18e), because any BRS invariant operator with the dimension 9/2 or less that contains  $c(x)$ ,  $\bar{c}(x)$  or  $B(x)$  is *always* BRS exact; the proof is given in Appendix B.

After the redefinition of the supercurrent,  $\mathcal{Z}_S \check{S}_\mu + \mathcal{Z}_T \check{T}_\mu \propto \check{S}_\mu^{\text{new}}$ ,<sup>14</sup> Eq. (4.20) takes the form of the conservation law of a supercurrent. Then the breaking

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<sup>13</sup>Since they rotate in opposite angles under  $U(1)_A$ , if the  $U(1)_A$  symmetry were exactly preserved, one could get rid of the possibility of  $\mathcal{O}_S(x)$ . The lattice regularization inevitably breaks the  $U(1)_A$ , however, to reproduce the axial anomaly.

<sup>14</sup>After this redefinition,  $\check{S}_\mu^{\text{new}}$  suffers from the superconformal (or gamma-trace) anomaly [61, 62, 63, 64, 65, 66, 67, 68, 69], because  $\gamma_\mu \check{T}_\mu(x) \neq 0$  while  $\gamma_\mu \check{S}_\mu(x) \equiv 0$ .

of SUSY due to the lattice regularization is characterized by the combination  $M - (1/a)\mathcal{Z}_\chi$ . One may remove this unphysical SUSY breaking by tuning the bare mass parameter  $M$  so that this combination vanishes:

$$M - \frac{1}{a}\mathcal{Z}_\chi \rightarrow 0. \quad (4.21)$$

We call this a supersymmetric limit [1] (a non-perturbative method to impose this condition has been proposed in Ref. [36]). It is considered that the chiral symmetric limit (3.13) coincides with this supersymmetric limit and thus condition (3.13) defines a unique supersymmetric theory [1]. For this, we must have

$$\mathcal{Z}_\chi = \frac{1}{2}\mathcal{Z}_P. \quad (4.22)$$

Eq. (4.20) shows that, however, the conservation law of the supercurrent suffers from an exotic breaking if  $\mathcal{Z}_{3F} \neq 0$ . If this breaking cannot be removed by a local counter term, this would imply an exotic SUSY anomaly which certainly what we do not expect in the present system. Even if this anomaly could be removed by a local counter term, its presence implies that, as Eq. (4.20) shows, tuning the mass parameter (4.21) alone will not lead to a supersymmetric theory, contradicting with what is believed. Thus we have to ensure that

$$\mathcal{Z}_{3F} = 0. \quad (4.23)$$

As noted above, however, it is not possible to conclude  $\mathcal{Z}_{3F} = 0$  by a simple symmetry argument alone. It appears that one needs some powerful machinery such as the WZ consistency condition [52]. The rest of this paper will be entirely devoted to the construction of the required consistency condition and its application to the proof of Eqs. (4.22) and (4.23).

## 5. Generalized BRS transformation in the lattice theory

### 5.1. General framework

We want to formulate a certain WZ consistency condition that constrains the quantum continuum limit of symmetry breaking terms attributed to the lattice regularization. Because of several reasons, unfortunately, this is a somewhat complicated task.

First, the algebraic structure of symmetry transformations is rather involved even in the continuum target theory. In 4D  $\mathcal{N} = 1$  SYM in the WZ

gauge, the algebra of SUSY transformations contains field-dependent gauge transformations. Moreover, the algebra closes only under the equation of motion of the (massless) gluino (i.e., on-shell closure). Thus, to construct a BRS-like nilpotent operation that contains SUSY (which will be a building block in the WZ consistency condition), one has to treat the SUSY, translation and gauge transformations on an equal footing. For this, we adopt the generalized BRS transformation developed in the continuum 4D  $\mathcal{N} = 1$  SYM in the WZ gauge. See Refs. [49, 50, 51] and references therein. The on-shell closure can also be incorporated into this framework by introducing a term that is quadratic in the “anti-field” of the gluino,  $K_\psi$  (see, for example, Refs. [70, 50]).

Second, we have a bare-mass term of the gluino that is inevitable for the tuning in lattice formulations; this term however explicitly breaks SUSY and the  $U(1)_A$  symmetry. If one wants to constrain the structure of radiative corrections by using the Slavnov–Taylor (ST) identity or the Zinn-Justin equation, those relations should not contain tree-level symmetry-breakings such as the gluino mass term. To circumvent this point, we introduce generalized BRS doublet fields (spurions)  $(u_V, v_V)$  and  $(u_A, v_A)$  which make the mass term “formally” BRS invariant [71, 51].

Finally, of course, SUSY and the infinitesimal translation transformations are not properly realized on lattice variables and, as the consequence, the nilpotency of the generalized BRS transformation on some of lattice variables is broken by  $O(a)$ . Then we have to carefully separate those  $O(a)$  breakings from the main part of the ST identity or the Zinn-Justin equation.

We thus define a generalized BRS transformation  $s$  in our lattice system

by

$$\begin{aligned}
sA_\mu(x) &\equiv [D_\mu c]^L(x) + [\bar{\xi}\gamma_\mu\psi]^L(x) - it_\nu\partial_\nu^S A_\mu(x), \\
s\psi(x) &\equiv -ig\{c(x), \psi(x)\} - \frac{1}{2}\sigma_{\mu\nu}\xi P_{\mu\nu}(x) - it_\mu\partial_\mu^S\psi(x) + i\theta\gamma_5\psi(x), \\
s\bar{\psi}(x) &\equiv -ig\{c(x), \bar{\psi}(x)\} + \frac{1}{2}\bar{\xi}\sigma_{\mu\nu}P_{\mu\nu}(x) - it_\mu\partial_\mu^S\bar{\psi}(x) + i\theta\bar{\psi}(x)\gamma_5, \\
sc(x) &\equiv -igc(x)^2 + \bar{\xi}\gamma_\mu\xi\frac{1}{2}[A_\mu(x) + A_\mu(x - a\hat{\mu})] - it_\mu\partial_\mu^S c(x), \\
s\bar{c}(x) &\equiv B(x) - it_\mu\partial_\mu^S\bar{c}(x), \\
sB(x) &\equiv \bar{\xi}\gamma_\mu\xi\partial_\mu^S\bar{c}(x) - it_\mu\partial_\mu^S B(x), \\
s\xi &\equiv i\theta\gamma_5\xi, \quad s\bar{\xi} = i\theta\bar{\xi}\gamma_5, \\
st_\mu &\equiv -i\bar{\xi}\gamma_\mu\xi, \\
s\theta &\equiv 0,
\end{aligned} \tag{5.1}$$

where  $\partial_\mu^S$  denotes the symmetric difference operator

$$\partial_\mu^S \equiv \frac{1}{2}(\partial_\mu + \partial_\mu^*). \tag{5.2}$$

In the above expressions, the gauge-ghost  $c(x)$ , anti-ghost  $\bar{c}(x)$  and the auxiliary field  $B(x)$  are common to the conventional lattice BRS transformation  $s_0$  (2.14), while  $\xi$ ,  $t_\mu$  and  $\theta$  are newly-introduced *constant* ghosts associated with the SUSY, translation and  $U(1)_A$  transformations, respectively.<sup>15</sup> These constant ghosts possess *opposite* Grassmann parity to the original transformation parameters; thus,  $\xi$  is Grassmann-even, and  $t_\mu$  and  $\theta$  are Grassmann-odd. The constant Grassmann-even ghost  $\xi$  is subject to the Majorana constraint,<sup>16</sup>

$$\bar{\xi} = \xi^T(-C^{-1}). \tag{5.3}$$

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<sup>15</sup>For these ghost variables, we use the same symbols as the corresponding classical transformation parameters; we think that no confusion will arise.

<sup>16</sup>It may seem strange that the generalized BRS transformation  $s$  (5.1) transforms  $c(x)$  and  $B(x)$  with the SUSY ghost  $\xi$ , though we assumed in Sec. 4 that these fields are singlets under the SUSY transformation. There is no contradiction, however, because the combination  $\bar{\xi}\gamma_\mu\xi$  *identically vanishes* when  $\xi$  is Grassmann-odd as the original SUSY parameter is. On the other hand, with such a combination with a Grassmann-even  $\xi$ ,  $s$  in the continuum limit becomes nilpotent (up to the equation of motion of the gluino).

Some useful identities that hold for such a Grassmann-even spinor are summarized in Appendix C. In the first relation of Eq. (5.1),  $[D_\mu c]^L(x)$  is given in Eq. (2.15) and  $[\bar{\xi}\gamma_\mu\psi]^L(x)$  is defined by

$$\begin{aligned} & [\bar{\xi}\gamma_\mu\psi]^L(x) \\ & \equiv \frac{1}{2}\bar{\xi}\gamma_\mu \left\{ \frac{iag\Delta_{A_\mu(x)}}{\exp[iag\Delta_{A_\mu(x)}] - 1} \psi(x) + \frac{iag\Delta_{A_\mu(x)}}{1 - \exp[-iag\Delta_{A_\mu(x)}]} \psi(x + a\hat{\mu}) \right\}. \end{aligned} \quad (5.4)$$

If there were only the gauge-ghost  $c(x)$  and the  $U(1)_A$  ghost  $\theta$  in Eq. (5.1), we would simply have  $s^2 = 0$  on all lattice variables, because algebras of the gauge and of the  $U(1)_A$  close on lattice variables. However, since the SUSY algebra (that contains the infinitesimal translation) is not properly realized on lattice variables, the nilpotency of  $s$  is broken by lattice artifacts. That is, on lattice variables, we have

$$s^2 A_\mu(x) = O(a), \quad (5.5a)$$

$$\begin{aligned} s^2 \psi(x) &= -\bar{\xi}\gamma_\mu \xi D_\mu \psi(x) + \sigma_{\mu\nu} \xi \bar{\xi} \gamma_\mu D_\nu \psi(x) + O(a) \\ &= \gamma_5 \xi \bar{\xi} \gamma_5 \not{D} \psi(x) + O(a) \\ &= \gamma_5 \xi \bar{\xi} \gamma_5 D \psi(x) + O(a), \end{aligned} \quad (5.5b)$$

$$s^2 c(x) = O(a), \quad (5.5c)$$

$$s^2 \bar{c}(x) = s^2 B(x) = s^2 \xi = s^2 t_\mu = s^2 \theta = 0. \quad (5.5d)$$

Moreover, as the right-hand side of Eq. (5.5b) shows,  $s^2 = 0$  on the gluino field  $\psi(x)$  holds only under the equation of motion of the massless gluino, even in the continuum theory [49, 50, 51]. (From the first line to the second line in Eq. (5.5b), we used the Fierz theorem in Eq. (A.1).)

We define the gauge-fixing term and the ghost-anti-ghost action with respect to the generalized BRS transformation  $s$  by

$$S_{\text{GF+FP}} \equiv -sa^4 \sum_x 2 \text{tr} \left\{ \bar{c}(x) \left[ \partial_\mu^* A_\mu(x) + \frac{\alpha}{2} B(x) \right] \right\}. \quad (5.6)$$

Note that this  $S_{\text{GF+FP}}$  reduces to our previous one  $S_{\text{GF+FP}}^{(0)}$  (2.13), when all new ghost variables,  $\xi$ ,  $t_\mu$  and  $\theta$ , vanish.

This is not the end of the story, however. To incorporate the mass term of the gluino that explicitly breaks SUSY and  $U(1)_A$ , we must further introduce

$s$ -doublet fields  $(u_V, v_V)$  and  $(u_A, v_A)$  following the procedure of Refs. [71, 51]:

$$\begin{aligned}
su_V(x) &\equiv v_V(x) + M - it_\mu \partial_\mu^S u_V(x) - 2i\theta u_A(x), \\
sv_V(x) &\equiv \bar{\xi} \gamma_\mu \xi \partial_\mu^S u_V(x) - it_\mu \partial_\mu^S v_V(x) - 2i\theta v_A(x), \\
su_A(x) &\equiv v_A(x) - it_\mu \partial_\mu^S u_A(x) - 2i\theta u_V(x), \\
sv_A(x) &\equiv \bar{\xi} \gamma_\mu \xi \partial_\mu^S u_A(x) - it_\mu \partial_\mu^S v_A(x) - 2i\theta [v_V(x) + M], \quad (5.7)
\end{aligned}$$

where  $u_V(x)$  and  $u_A(x)$  are Grassmann-odd, while  $v_V(x)$  and  $v_A(x)$  are Grassmann-even. On these variables, we have an exact nilpotency of  $s$ :

$$s^2 u_V(x) = s^2 u_A(x) = s^2 v_V(x) = s^2 v_A(x) = 0. \quad (5.8)$$

Using these fields, we define a “generalized” mass term by

$$\begin{aligned}
S_{\text{mass}} &= -sa^4 \sum_x \text{tr} \{ \bar{\psi}(x) [u_V(x) + u_A(x)\gamma_5] \psi(x) \} \\
&= a^4 \sum_x M \text{tr} [\bar{\psi}(x)\psi(x)] \\
&\quad + a^4 \sum_x \text{tr} \{ \bar{\psi}(x) [v_V(x) + v_A(x)\gamma_5] \psi(x) \} \\
&\quad - a^4 \sum_x \text{tr} \{ \bar{\xi} \sigma_{\mu\nu} P_{\mu\nu}(x) [u_V(x) + u_A(x)\gamma_5] \psi(x) \} \\
&\quad - a^4 \sum_x (-i)t_\mu \tilde{\partial}_\mu^S \text{tr} \{ \bar{\psi}(x) [u_V(x) + u_A(x)\gamma_5] \psi(x) \}, \quad (5.9)
\end{aligned}$$

where  $\tilde{\partial}_\mu^S$  is defined by the Leibniz rule,

$$\tilde{\partial}_\mu^S(X_1 \dots X_n) \equiv (\partial_\mu^S X_1) \dots X_n + \dots + X_1 \dots X_{n-1} (\partial_\mu^S X_n). \quad (5.10)$$

Again,  $S_{\text{mass}}$  reduces to our original mass term  $S_{\text{mass}}^{(0)}$  (2.6) when the newly-introduced fields,  $(u_V, v_V)$  and  $(u_A, v_A)$ , vanish.

We also introduce source terms for lattice dynamical variables

$$\begin{aligned}
S_{\text{source1}} &\equiv a^4 \sum_x [J_{A\mu}^a(x) A_\mu^a(x) + \bar{J}_\psi^a(x) \psi^a(x) + J_c^a(x) c^a(x) \\
&\quad + J_{\bar{c}}^a(x) \bar{c}^a(x) + J_B^a(x) B^a(x)] \quad (5.11)
\end{aligned}$$

(we may define  $J_\psi^a(x)$  by  $\bar{J}_\psi^a(x) \equiv J_\psi^{Ta}(x)(-C^{-1})$ ) and, following the standard procedure (see, for example, Ref. [72]), source terms associated with the generalized BRS transformation  $s$ ,

$$S_{\text{source2}} \equiv a^4 \sum_x [K_{A\mu}^a(x) s A_\mu^a(x) + \bar{K}_\psi^a(x) s \psi^a(x) + K_c^a(x) s c^a(x)] - a^4 \sum_x \frac{1}{2} \bar{K}_\psi^a(x) \gamma_5 \xi \bar{K}_\psi^a(x) \gamma_5 \xi. \quad (5.12)$$

(Again we may define  $K_\psi^a(x)$  by  $\bar{K}_\psi^a(x) \equiv K_\psi^{Ta}(x)(-C^{-1})$ .) For the massive theory, it turns out that we need one more term,

$$S_\Delta \equiv a^4 \sum_x \bar{K}_\psi^a(x) \gamma_5 \xi \bar{\psi}^a(x) [u_V(x) + u_A(x) \gamma_5] \gamma_5 \xi - a^4 \sum_x \frac{1}{2} \bar{\psi}^a(x) [u_V(x) + u_A(x) \gamma_5] \gamma_5 \xi \bar{\psi}^a(x) [u_V(x) + u_A(x) \gamma_5] \gamma_5 \xi. \quad (5.13)$$

The last term of Eq. (5.12), which is quadratic in the source  $K_\psi(x)$ , and  $S_\Delta$  are required to make the following ST identity or the Zinn-Justin equation hold off-shell up to  $O(a)$  lattice artifacts.

Thus, our generalized total lattice action is

$$S_{\text{tot}} \equiv S_{\text{gluon}} + S_{\text{gluino}} + S_{\text{GF+FP}} + S_{\text{mass}} + S_{\text{source1}} + S_{\text{source2}} + S_\Delta. \quad (5.14)$$

Using this, we define the generating functional for the connected diagram  $W$ ,

$$e^{-W[J,K,\xi,t,\theta,u,v]} \equiv \int d\mu e^{-S_{\text{tot}}}, \quad (5.15)$$

where  $d\mu$  denotes the integration measure for dynamical variables,  $U(x, \mu)$ ,  $\psi(x)$ ,  $c(x)$ ,  $\bar{c}(x)$  and  $B(x)$ . An important point to recognize here is that those newly-introduced variables,  $\xi$ ,  $t_\mu$ ,  $\theta$ ,  $u_V(x)$ ,  $u_A(x)$ ,  $v_V(x)$  and  $v_A(x)$ , are all *non-dynamical*, as the argument of the above  $W$  indicates. One can always set those external variables zero and then the system reduces to our original lattice theory in Sec. 2. In this way, the dynamics of the original system can always be reproduced; yet, those new variables are quite useful to organize the underlying symmetry structure.

For  $\mathcal{P}$  and  $\mathcal{T}$  transformation properties of new variables, we can set

$$\begin{aligned}\xi &\xrightarrow{\mathcal{P}} i\gamma_0\xi, & \bar{\xi} &\xrightarrow{\mathcal{P}} -i\bar{\xi}\gamma_0, & t_0 &\xrightarrow{\mathcal{P}} t_0, & t_k &\xrightarrow{\mathcal{P}} -t_k, & \theta &\xrightarrow{\mathcal{P}} -\theta, \\ u_V(x_0, \vec{x}) &\xrightarrow{\mathcal{P}} u_V(x_0, -\vec{x}), & v_V(x_0, \vec{x}) &\xrightarrow{\mathcal{P}} v_V(x_0, -\vec{x}), \\ u_A(x_0, \vec{x}) &\xrightarrow{\mathcal{P}} -u_A(x_0, -\vec{x}), & v_A(x_0, \vec{x}) &\xrightarrow{\mathcal{P}} -v_A(x_0, -\vec{x}),\end{aligned}\quad (5.16)$$

and

$$\begin{aligned}\xi &\xrightarrow{\mathcal{T}} i\gamma_0\gamma_5\xi, & \bar{\xi} &\xrightarrow{\mathcal{T}} -i\bar{\xi}\gamma_5\gamma_0, & t_0 &\xrightarrow{\mathcal{T}} -t_0, & t_k &\xrightarrow{\mathcal{T}} t_k, & \theta &\xrightarrow{\mathcal{T}} -\theta, \\ u_V(x_0, \vec{x}) &\xrightarrow{\mathcal{T}} u_V(-x_0, \vec{x}), & v_V(x_0, \vec{x}) &\xrightarrow{\mathcal{T}} v_V(-x_0, \vec{x}), \\ u_A(x_0, \vec{x}) &\xrightarrow{\mathcal{T}} -u_A(-x_0, \vec{x}), & v_A(x_0, \vec{x}) &\xrightarrow{\mathcal{T}} -v_A(-x_0, \vec{x}),\end{aligned}\quad (5.17)$$

so that  $s$  in Eqs. (5.1) and (5.7) preserves transformation properties under  $\mathcal{P}$  and  $\mathcal{T}$ ; for example, one sees that  $[\bar{\xi}\gamma_\mu\psi]^L(x)$  transforms in an identical way as  $A_\mu(x)$ . Then, it is easy to find appropriate  $\mathcal{P}$  and  $\mathcal{T}$  transformations of source fields  $J$  and  $K$  such that the total action  $S_{\text{tot}}$  (5.14) is invariant under  $\mathcal{P}$  and  $\mathcal{T}$ .

Now, a crucial property of the above total action  $S_{\text{tot}}$ , that can be obtained by a careful calculation using relations in Appendix C, is

$$\begin{aligned}sS_{\text{tot}} - a^4 \sum_x \bar{K}'^a_\psi(x) \gamma_5 \xi \bar{\xi} \gamma_5 \frac{\delta}{\delta \bar{\psi}^a(x)} S_{\text{tot}} \\ = a^4 \sum_x \left\{ J_{A_\mu}^a(x) s A_\mu^a(x) - \bar{J}_\psi^a(x) [s\psi^a(x) - \gamma_5 \xi \bar{K}'^a_\psi(x) \gamma_5 \xi] \right. \\ \left. - J_c^a(x) s c^a(x) - J_c^a(x) s \bar{c}^a(x) + J_B^a(x) s B^a(x) \right\} \\ + a^4 \sum_x [\bar{\xi} X_S(x) + \theta X_A(x)] + \bar{c} \cdot \mathcal{B}_{\bar{c}} + K' \cdot \mathcal{B}_{K'} + t \cdot \mathcal{B}_t,\end{aligned}\quad (5.18)$$

where

$$\bar{K}'_\psi(x) \equiv \bar{K}_\psi(x) - \bar{\psi}(x) [u_V(x) + u_A(x) \gamma_5]. \quad (5.19)$$

In deriving Eq. (5.18), we parametrized the breaking of the super,  $U(1)_A$  and translation symmetries in our original lattice action as

$$s(S_{\text{gluon}} + S_{\text{gluino}}) \equiv a^4 \sum_x [\bar{\xi} X_S(x) + i\theta X_A(x) + t_\mu X_{t\mu}(x)]. \quad (5.20)$$



This  $X_S$  is thus *identical* to our previous definition in Eq. (4.9) and  $X_A$  is also *identical* to  $X_A$  in Eq. (3.2); this correspondence will be the key to our analysis below. Since the continuum action without the gluon mass term possesses the super,  $U(1)_A$  and translation symmetries, all these breakings,  $X_S(x)$ ,  $X_A(x)$  and  $X_{t\mu}(x)$ , are of  $O(a)$ .

Explicit forms of the last three combinations in Eq. (5.18) are given by

$$\bar{c} \cdot \mathcal{B}_{\bar{c}} \equiv -a^4 \sum_x \bar{c}^a(x) \partial_\mu^* s^2 A_\mu^a(x), \quad (5.21)$$

$$\begin{aligned} K' \cdot \mathcal{B}_{K'} \equiv a^4 \sum_x \bigg\{ & -K_{A_\mu}^a(x) s^2 A_\mu^a(x) + \bar{K}_\psi'^a(x) [s^2 \psi^a(x) - \gamma_5 \xi \bar{\xi} \gamma_5 D \psi^a(x)] \\ & + K_c^a(x) s^2 c^a(x) \bigg\}, \end{aligned} \quad (5.22)$$

and

$$\begin{aligned} t \cdot \mathcal{B}_t & \equiv a^4 \sum_x t_\mu X_{t_\mu}(x) \\ & - a^4 \sum_x (-i) t_\mu \bar{K}_\psi'^a(x) \gamma_5 \xi \left( \partial_\mu^S \bar{K}_\psi^a(x) \right. \\ & \quad \left. - \tilde{\partial}_\mu^S \{ \bar{\psi}^a(x) [u_V(x) + u_A(x) \gamma_5] \} \right) \gamma_5 \xi \\ & - a^4 \sum_x (-i) t_\mu \bar{K}_\psi'^a(x) \gamma_5 \xi \left( \partial_\mu^S - \tilde{\partial}_\mu^S \right) \{ \bar{\psi}^a(x) [u_V(x) + u_A(x) \gamma_5] \gamma_5 \xi \}. \end{aligned} \quad (5.23)$$

It is obvious that these three combinations are of  $O(a)$ , especially in view of Eq. (5.5). Another crucial property of  $S_{\text{tot}}$  to note is

$$\frac{\delta}{\delta \bar{K}_\psi^a(x)} S_{\text{tot}} = s \psi^a(x) - \gamma_5 \xi \bar{K}_\psi'^a(x) \gamma_5 \xi. \quad (5.24)$$

## 5.2. Generalized ST identity or Zinn-Justin equation and an associated WZ consistency condition

Now, the combination in the left-hand side of relation (5.18) prompts us to consider the following infinitesimal change of variables for the functional integral (5.15). Letting  $\varepsilon$  be a Grassmann-odd parameter,

$$\Phi(x) \rightarrow \Phi(x) + \delta \Phi(x), \quad \delta \Phi(x) = \varepsilon s \Phi(x), \quad (5.25)$$

for all variables  $\Phi$  *including* external ones,  $\xi$ ,  $t_\mu$ ,  $\theta$ ,  $u_V(x)$ ,  $u_A(x)$ ,  $v_V(x)$  and  $v_A(x)$ , except the gluino field  $\psi(x)$ , for which,

$$\psi(x) \rightarrow \psi(x) + \delta\psi(x), \quad \delta\psi(x) = \varepsilon [s\psi(x) - \gamma_5 \xi \bar{K}'_\psi(x) \gamma_5 \xi]. \quad (5.26)$$

Although the last term depends on  $\psi(x)$  itself through the definition of  $K'_\psi(x)$  (5.19), the Jacobian  $J$  associated with this infinitesimal change of variable is unity, because

$$\ln J = \varepsilon \sum_x \bar{\xi} [u_V(x) + u_A(x) \gamma_5] \xi = 0, \quad (5.27)$$

with the lattice regularization (note Eq. (C.1)).

From the above change of variables, using Eq. (5.18), we have the identity

$$\begin{aligned} & \left\{ s\xi \frac{\partial}{\partial \xi} + st_\mu \frac{\partial}{\partial t_\mu} + s\theta \frac{\partial}{\partial \theta} \right. \\ & \quad + a^4 \sum_x \left[ su_V(x) \frac{\delta}{\delta u_V(x)} + su_A(x) \frac{\delta}{\delta u_A(x)} \right. \\ & \quad \quad \left. \left. + sv_V(x) \frac{\delta}{\delta v_V(x)} + sv_A(x) \frac{\delta}{\delta v_A(x)} \right] \right\} W[J, K, \xi, t, \theta, u, v] \\ &= \left\langle a^4 \sum_x \left\{ J_{A_\mu}^a(x) sA_\mu^a(x) - \bar{J}_\psi^a(x) [s\psi^a(x) - \gamma_5 \xi \bar{K}'_\psi(x) \gamma_5 \xi] \right. \right. \\ & \quad \left. \left. - J_c^a(x) sc^a(x) - J_{\bar{c}}^a(x) s\bar{c}^a(x) + J_B^a(x) sB^a(x) \right\} \right. \\ & \quad \left. + a^4 \sum_x [\bar{\xi} X_S(x) + i\theta X_A(x)] + \bar{c} \cdot \mathcal{B}_{\bar{c}} + K' \cdot \mathcal{B}_{K'} + t \cdot \mathcal{B}_t \right\rangle_{J, K, \xi, t, \theta, u, v}. \end{aligned} \quad (5.28)$$

As usual, we define the 1PI effective action  $\Gamma$  by the Legendre transform of  $W$  only with respect to the source  $J$ :

$$\begin{aligned} & \Gamma[A_\mu, \psi, c, \bar{c}, B; K, \xi, t, \theta, u, v] \\ & \equiv W[J, K, \xi, t, \theta, u, v] \\ & \quad - a^4 \sum_x [J_{A_\mu}^a(x) A_\mu^a(x) + \bar{J}_\psi^a(x) \psi^a(x) + J_c^a(x) c^a(x) \\ & \quad \quad + J_{\bar{c}}^a(x) \bar{c}^a(x) + J_B^a(x) B^a(x)]. \end{aligned} \quad (5.29)$$

Here and in what follows, we use the same symbols for the integration variables and the expectation values for notational simplicity; for example,  $A_\mu^a(x)$  in the argument of the effective action  $\Gamma$  actually means  $\langle A_\mu^a(x) \rangle_{J,K,\xi,t,\theta,u,v}$ .

Then noting relations such as,

$$\frac{\partial}{\partial \xi} W = \frac{\partial \Gamma}{\partial \xi}, \quad (5.30)$$

$$J_\mu^a(x) = -\frac{\delta \Gamma}{\delta A_\mu^a(x)}, \quad (5.31)$$

$$\langle s\psi^a(x) - \gamma_5 \xi \bar{K}_\psi'^a(x) \gamma_5 \xi \rangle_{J,K,\xi,t,\theta,u,v} = \frac{\delta \Gamma}{\delta \bar{K}_\psi^a(x)} \quad (5.32)$$

(the last one follows from Eq. (5.24)) etc., we can express identity (5.28) in terms of the effective action  $\Gamma$ . This gives rise to the following identity (the ST identity or the Zinn-Justin equation in the present context):

$$\mathcal{S}(\Gamma) = \left\langle a^4 \sum_x [\bar{\xi} X_S(x) + i\theta X_A(x)] + \bar{c} \cdot \mathcal{B}_{\bar{c}} + K' \cdot \mathcal{B}_{K'} + t \cdot \mathcal{B}_t \right\rangle_{J,K,\xi,t,\theta,u,v}, \quad (5.33)$$

where the combination  $\mathcal{S}(F)$  is defined for an arbitrary functional  $F$  by,

$$\begin{aligned} \mathcal{S}(F) \equiv & a^4 \sum_x \left[ \frac{\delta F}{\delta K_{A_\mu}^a(x)} \frac{\delta F}{\delta A_\mu^a(x)} + \frac{\delta F}{\delta \bar{K}_\psi^a(x)} \frac{\delta F}{\delta \psi^a(x)} + \frac{\delta F}{\delta K_c^a(x)} \frac{\delta F}{\delta c^a(x)} \right] \\ & + a^4 \sum_x \left[ s\bar{c}^a(x) \frac{\delta F}{\delta \bar{c}^a(x)} + sB^a(x) \frac{\delta F}{\delta B^a(x)} \right] \\ & + s\xi \frac{\partial F}{\partial \xi} + st_\mu \frac{\partial F}{\partial t_\mu} + s\theta \frac{\partial F}{\partial \theta} \\ & + a^4 \sum_x \left[ su_V(x) \frac{\delta F}{\delta u_V(x)} + su_A(x) \frac{\delta F}{\delta u_A(x)} \right. \\ & \quad \left. + sv_V(x) \frac{\delta F}{\delta v_V(x)} + sv_A(x) \frac{\delta F}{\delta v_A(x)} \right]. \quad (5.34) \end{aligned}$$

Corresponding to this combination, we introduce an operation  $\mathcal{D}(F)$  by

$$\begin{aligned}
\mathcal{D}(F) \equiv & a^4 \sum_x \left[ \frac{\delta F}{\delta A_\mu^a(x)} \frac{\delta}{\delta K_{A_\mu}^a(x)} + \frac{\delta F}{\delta K_{A_\mu}^a(x)} \frac{\delta}{\delta A_\mu^a(x)} \right. \\
& + \frac{\delta F}{\delta \bar{K}_\psi^a(x)} \frac{\delta}{\delta \psi^a(x)} + \frac{\delta F}{\delta \psi^a(x)} \frac{\delta}{\delta \bar{K}_\psi^a(x)} \\
& \left. + \frac{\delta F}{\delta K_c^a(x)} \frac{\delta}{\delta c^a(x)} + \frac{\delta F}{\delta c^a(x)} \frac{\delta}{\delta K_c^a(x)} \right] \\
& + a^4 \sum_x \left[ s \bar{c}^a(x) \frac{\delta}{\delta \bar{c}^a(x)} + s B^a(x) \frac{\delta}{\delta B^a(x)} \right] \\
& + s \xi \frac{\partial}{\partial \xi} + s t_\mu \frac{\partial}{\partial t_\mu} + s \theta \frac{\partial}{\partial \theta} \\
& + a^4 \sum_x \left[ s u_V(x) \frac{\delta}{\delta u_V(x)} + s u_A(x) \frac{\delta}{\delta u_A(x)} \right. \\
& \left. + s v_V(x) \frac{\delta}{\delta v_V(x)} + s v_A(x) \frac{\delta}{\delta v_A(x)} \right]. \quad (5.35)
\end{aligned}$$

Then, for an *arbitrary* (Grassmann-even) functional  $F$ , we have

$$\mathcal{D}(F) \mathcal{S}(F) = 0. \quad (5.36)$$

As one can verify straightforwardly, this relation follows solely from the nilpotency  $s^2 = 0$  on the variables,  $\bar{c}(x)$ ,  $B(x)$ ,  $\xi$ ,  $t_\mu$ ,  $\theta$ ,  $u_V(x)$ ,  $u_A(x)$ ,  $v_V(x)$  and  $v_A(x)$ ; recall Eqs. (5.5d) and (5.8). Although the nilpotency of  $s$  on the variables,  $A_\mu(x)$ ,  $\psi(x)$  and  $c(x)$ , is broken by the lattice regularization, the nilpotency on these variables is not necessary to derive Eq. (5.36); this is a crucial observation.

Since Eq. (5.36) holds for arbitrary  $F$ , it holds of course for  $\Gamma$ . Then, combined with Eq. (5.33), Eq. (5.36) provides a strong constraint on the possible form of the breaking terms, the right-hand side of Eq. (5.33). That is, we have

$$\mathcal{D}(\Gamma) \left\langle a^4 \sum_x [\bar{\xi} X_S(x) + i \theta X_A(x)] + \bar{c} \cdot \mathcal{B}_{\bar{c}} + K' \cdot \mathcal{B}_{K'} + t \cdot \mathcal{B}_t \right\rangle_{J,K,\xi,t,\theta,u,v} = 0. \quad (5.37)$$

We now expand the effective action  $\Gamma$  in the powers of the loop-counting parameter  $\hbar$  as  $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2 + \dots$ , where  $\Gamma_0$  is the tree level action,

$$\Gamma_0 = S_{\text{tot}}. \quad (5.38)$$

In the continuum limit, the expectation value in Eq. (5.37) is  $O(\hbar)$  or higher because the  $O(a)$  breakings attributed to the lattice regularization can survive only through radiative corrections. Thus, taking the  $O(\hbar^n)$  ( $n \geq 1$ ) term of Eq. (5.37), where  $n$  is the lowest order of the loop expansion in which the expectation value does not vanish, we have

$$\begin{aligned} \mathcal{D}(S_{\text{tot}}) \left\langle a^4 \sum_x [\bar{\xi} X_S(x) + i\theta X_A(x)] + \bar{c} \cdot \mathcal{B}_{\bar{c}} + K' \cdot \mathcal{B}_{K'} + t \cdot \mathcal{B}_t \right\rangle_{J,K,\xi,t,\theta,u,v}^{O(\hbar^n)} \\ = 0. \end{aligned} \quad (5.39)$$

This is the WZ consistency condition that we were seeking.

## 6. Simplified consistency condition and its application

### 6.1. Simplified consistency condition

To simplify the analysis, we consider the consistency condition (5.39) for a special configuration of expectation values and external variables. First, we assume that expectation values,  $A_\mu(x)$ ,  $\psi(x)$  and  $c(x)$ , satisfy the equations of motion of the total action  $S_{\text{tot}}$ . This removes the functional derivative with respect to  $K$  from the expression for  $\mathcal{D}(S_{\text{tot}})$  (see Eq. (5.35)) and  $\mathcal{D}(S_{\text{tot}})$  simply becomes the  $s$  transformation. We further set

$$\bar{c}(x) = B(x) = 0, \quad (6.1)$$

$$K_{A_\mu}(x) = K_\psi(x) = K_c(x) = t_\mu = u_V(x) = u_A(x) = v_V(x) = v_A(x) = 0. \quad (6.2)$$

We can now relate Eqs. (3.9) and (4.17) to Eq. (5.39). Since we assumed in Eq. (6.2) that all  $K$  vanish, expectation values in Eq. (5.39) are computed without the source term for the composite operators,  $S_{\text{source2}}$  (5.12); we have only the source term for the elementary fields,  $S_{\text{source1}}$  (5.11). In the language of Eqs. (3.6) and (4.13), this situation corresponds to the situation that the inserted operator  $\mathcal{O}(y, z, \dots)$  is a collection of *elementary fields*. In this situation, then, we can neglect the  $O(a)$  terms in Eqs. (3.9e) and (4.17h), because

no  $O(1/a)$  divergence can arise in correlation functions of a renormalized operator and elementary fields. Moreover, since  $O(\hbar^n)$  is the lowest non-trivial order of the loop expansion in which the expectation value becomes non-zero, we can set, for example,  $\langle \mathcal{Z}_T \partial_\mu \check{T}_\mu(x) \rangle^{O(\hbar^n)} = \mathcal{Z}_T^{O(\hbar^n)} \langle \partial_\mu \check{T}_\mu(x) \rangle^{O(\hbar^0)}$ . The expectation values of operators in Eqs. (3.9) and (4.17) are thus evaluated in the tree level approximation and expressions in Eqs. (3.9) and (4.17) can be regarded as the expectation values themselves.

In this way, in the continuum limit, consistency condition (5.39) is simplified to

$$\begin{aligned} & \int d^4x \left[ i\theta \bar{\xi} \gamma_5 X_S(x)^{O(\hbar^n)} + \bar{\xi} r X_S(x)^{O(\hbar^n)} - i\theta r X_A(x)^{O(\hbar^n)} \right] \\ & + \int d^4x M \langle \bar{\psi}^a(x) [s^2 \psi^a(x) - \gamma_5 \bar{\xi} \gamma_5 D \psi^a(x)] \rangle^{O(\hbar^n)} \\ & + \int d^4x (-i) \bar{\xi} \gamma_\mu \xi \langle X_{t_\mu}(x) \rangle^{O(\hbar^n)} = 0, \end{aligned} \quad (6.3)$$

in terms of expressions in Eqs. (3.9) and (4.17) (here we should not include the  $O(a)$  terms, Eqs. (3.9e) and (4.17h)), where

$$\begin{aligned} r A_\mu(x) &\equiv D_\mu c(x) + \bar{\xi} \gamma_\mu \psi(x), \\ r \psi(x) &\equiv -ig \{c(x), \psi(x)\} - \frac{1}{2} \sigma_{\mu\nu} \xi F_{\mu\nu}(x) + i\theta \gamma_5 \psi(x), \\ r \bar{\psi}(x) &\equiv -ig \{c(x), \bar{\psi}(x)\} + \frac{1}{2} \bar{\xi} \sigma_{\mu\nu} F_{\mu\nu}(x) + i\theta \bar{\psi}(x) \gamma_5, \\ r c(x) &\equiv -ig c(x)^2 + \bar{\xi} \gamma_\mu \xi A_\mu(x), \\ r \xi &\equiv i\theta \gamma_5 \xi, \quad r \bar{\xi} = i\theta \bar{\xi} \gamma_5, \\ r t_\mu &\equiv -i \bar{\xi} \gamma_\mu \xi, \\ r u_V(x) &\equiv M, \\ r v_A(x) &\equiv -2i\theta M, \end{aligned} \quad (6.4)$$

and  $r \equiv 0$  on other variables. Note that the  $s^2 \psi^a$ -term in Eq. (6.3) is quadratic in  $\xi$ , i.e.,  $O(\xi^2)$ .

## 6.2. $O(\theta^1, \xi^0)$ term

Relation (6.3) must hold order by order in  $\theta$  and  $\xi$ . Let us first consider the  $O(\theta^1, \xi^0)$  term. This gives

$$\int d^4x \left[ r X_A(x)^{O(\hbar^n)} \right]_{O(\theta^0, \xi^0)} = 0. \quad (6.5)$$

When  $\theta = \xi = 0$ ,  $r$  is nothing but the conventional gauge BRS transformation in the continuum theory. Thus, Eq. (6.5) simply tells that  $X_A(x)$  must be (gauge) BRS invariant up to a total divergence.<sup>17</sup> This is trivially true for Eqs. (3.9a)–(3.9c), but it constrains the possible structure of the BRS non-invariant piece, Eq. (3.9d).

### 6.3. $O(\theta^0, \xi^1)$ term

Similarly, we have from the  $O(\theta^0, \xi^1)$  term of Eq. (6.3),

$$\int d^4x \left[ r X_S(x)^{O(h^n)} \right]_{O(\theta^0, \xi^0)} = 0. \quad (6.6)$$

As an application of this relation, let us ask whether a BRS non-invariant combination

$$\gamma_\mu \operatorname{tr} [\psi(x) \{A_\nu(x), F_{\mu\nu}(x)\}], \quad (6.7)$$

can appear in Eq. (4.17g) or not. For this operator, we have

$$\begin{aligned} & [r \gamma_\mu \operatorname{tr} [\psi(x) \{A_\nu(x), F_{\mu\nu}(x)\}]]_{O(\theta^0, \xi^0)} \\ &= -\gamma_\mu \operatorname{tr} [\psi(x) \{\partial_\nu c(x), F_{\mu\nu}(x)\}], \end{aligned} \quad (6.8)$$

which is proportional to the totally-symmetric tensor

$$d^{abc} \equiv \operatorname{tr} (T^a \{T^b, T^c\}). \quad (6.9)$$

Then the question is that whether Eq. (6.8) can be made into a total-divergence by adding the  $r$ -transformation of some operator other than minus Eq. (6.7). We see that this is impossible as follows.

Note first that except the term  $\partial_\mu c(x)$  in  $rA_\mu(x)$  of Eq. (6.4),  $c(x)$  in the  $r$ -operation appears in the adjoint action. Since the adjoint action is proportional to the antisymmetric structure constant, the adjoint action with the ghost  $c(x)$  cannot produce the totally-symmetric tensor  $d^{abc}$ . The only way to form  $d^{abc}$  is to use the  $\partial_\mu c(x)$  in  $rA_\mu(x)$ . Thus the generic operator

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<sup>17</sup>Thus,  $X_A(x)$  (and  $X_S(x)$  too) is an element of the local (gauge) BRS cohomology (see, for example, Ref. [73]). If one had a complete classification of the cohomology on the basis of lattice symmetries alone (e.g., the hypercubic symmetry instead of the Lorentz symmetry), it would greatly simplify the following discussion.

to be added to Eq. (6.8) must have the following structure,<sup>18</sup>

$$\begin{aligned}
& - [r\gamma_\mu \{C_{\mu\nu\rho\sigma} \text{tr} [\psi(x) \{A_\nu(x), F_{\rho\sigma}(x)\}] \\
& \quad + D_{\mu\nu\rho\sigma} \text{tr} [\psi(x) \{A_\nu(x), \partial_\rho A_\sigma(x)\}]\}]_{O(\theta^0, \xi^0)} \\
& = \gamma_\mu \{C_{\mu\nu\rho\sigma} \text{tr} [\psi(x) \{\partial_\nu c(x), F_{\rho\sigma}(x)\}] \\
& \quad + D_{\mu\nu\rho\sigma} \text{tr} [\psi(x) \{\partial_\nu c(x), \partial_\rho A_\sigma(x)\}] \\
& \quad + D_{\mu\nu\rho\sigma} \text{tr} [\psi(x) \{A_\nu(x), \partial_\rho \partial_\sigma c(x)\}]\}. \tag{6.10}
\end{aligned}$$

For the sum of this and Eq. (6.8) to become a total divergence, terms proportional to  $\text{tr}[\psi(x) \{\partial_\nu c(x) [A_\rho(x), A_\sigma(x)]\}]$  should cancel out (because it cannot be a total divergence). This requires  $C_{\mu\nu\rho\sigma} = \delta_{\mu\rho} \delta_{\nu\sigma}$ . However, then the first term in Eq. (6.10) is just the  $r$ -transformation of minus Eq. (6.7). This shows that Eq. (6.7) cannot appear in the BRS non-invariant piece in  $X_S(x)$  (4.17g); this fact has an important implication below.

#### 6.4. $O(\theta^1, \xi^1)$ terms

At long last, we are now ready to attack one of our main problems. For this, we take the  $O(\theta^1, \xi^1)$  terms of Eq. (6.3):

$$\begin{aligned}
& \int d^4x \left\{ i\theta \bar{\xi} \gamma_5 X_S(x)_{O(\theta^0, \xi^0)}^{O(h^n)} + \bar{\xi} [r X_S(x)_{O(\theta^1, \xi^0)}^{O(h^n)} \right. \\
& \quad \left. - i\theta [r X_A(x)_{O(\theta^0, \xi^1)}^{O(h^n)}] \right\} = 0. \tag{6.11}
\end{aligned}$$

Then, substituting Eqs. (3.9a)–(3.9c) and Eqs. (4.17a)–(4.17e) in Eq. (6.11), we have

$$\mathcal{Z}_X^{O(h^n)} - \frac{1}{2} \mathcal{Z}_P^{O(h^n)} = 0, \tag{6.12}$$

as the coefficient of the combination

$$-\frac{1}{a} \int d^4x \, 2i\theta \bar{\xi} \gamma_5 \sigma_{\mu\nu} \text{tr} [\psi(x) F_{\mu\nu}(x)]. \tag{6.13}$$

It is easy to see that the terms containing  $c(x)$ ,  $\bar{c}(x)$  or  $B(x)$ , Eq. (4.17f), cannot contribute to Eq. (6.13). We can see also that BRS non-invariant terms in Eqs. (3.9d) and (4.17g) do not contribute to Eq. (6.13) as follows.

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<sup>18</sup>There exists another possible combination  $-r\gamma_\mu E_{\mu\nu\rho\sigma} \text{tr} [\partial_\rho \psi(x) \{A_\nu(x), A_\sigma(x)\}]$ , but this is reduced to other combinations in Eq. (6.10) up to a total divergence.



First, since the  $O(\theta)$  term in Eq. (6.4) does not change the gauge index structure and Eq. (6.13) is proportional to  $\delta^{ab} = 2 \text{tr}(T^a T^b)$ , the possible contribution from Eq. (4.17g) to Eq. (6.13) comes from

$$-\frac{1}{a}2\bar{\xi}\left[r\left\{2F_{\mu\nu\rho\sigma}\sigma_{\mu\nu}\text{tr}[\psi(x)\partial_\rho A_\sigma(x)]\right.\right. \\ \left.\left.+G_{\mu\nu\rho\sigma}\sigma_{\mu\nu}\text{tr}[\psi(x)ig[A_\rho(x),A_\sigma(x)]]\right\}\right]_{O(\theta^1,\xi^0)}. \quad (6.14)$$

However, for (the integral of) this to coincide with Eq. (6.13) up to a total divergence, we have to have  $G_{\mu\nu\rho\sigma} = c\delta_{\mu\rho}\delta_{\nu\sigma}$ , because the combination  $\text{tr}[\psi(x)[A_\rho(x),A_\sigma(x)]]$  does not contain any derivative. Once this becomes the case, then, we have to moreover have  $F_{\mu\nu\rho\sigma} = c\delta_{\mu\rho}\delta_{\nu\sigma}$ . Then the combination in the curly brackets of Eq. (6.14) is  $c\sigma_{\mu\nu}\text{tr}[\psi(x)F_{\mu\nu}(x)]$ , the operator  $(1/a)c\bar{\chi}(x)$  in Eq. (4.17c) that should not be contained in Eq. (4.17g). On the other hand, concerning Eq. (3.9d), the possible combination that could contribute to Eq. (6.13) is

$$-i\theta\left[r\left\{\frac{1}{a}H_{\mu\rho\sigma}\text{tr}[A_\mu(x)F_{\rho\sigma}(x)]+\frac{1}{a}I_{\mu\rho\sigma}2\text{tr}[A_\mu(x)\partial_\rho A_\sigma(x)]\right\}\right]_{O(\theta^0,\xi^1)}. \quad (6.15)$$

However, this cannot contribute to Eq. (6.13), because  $rA_\mu(x)$  (6.4) produces  $\bar{\xi}\gamma_\mu\psi(x)$ , but this combination is linearly independent of  $\bar{\xi}\gamma_5\sigma_{\mu\nu}\psi(x)$  (the coefficient of Eq. (6.13)).<sup>19</sup> By this way, we see that Eqs. (3.9d) and (4.17g) do not contribute to combination (6.13).

Besides these, in the present case, we have to take into consideration also the possible modification of Eqs. (3.9) and (4.17) by the presence of newly-introduced external variables. From Eqs. (6.4), we see that if there exist mixings such that

$$X_S(x) \sim -\frac{1}{a}\frac{u_V(x)}{M}2i\theta\gamma_5\sigma_{\mu\nu}\text{tr}[\psi(x)F_{\mu\nu}(x)], \\ X_S(x) \sim \frac{1}{a}\frac{v_A(x)}{M}\gamma_5\sigma_{\mu\nu}\text{tr}[\psi(x)F_{\mu\nu}(x)], \\ X_A(x) \sim \frac{1}{a}\frac{u_V(x)}{M}2\bar{\xi}\gamma_5\sigma_{\mu\nu}\text{tr}[\psi(x)F_{\mu\nu}(x)], \quad (6.16)$$

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<sup>19</sup>This is because the former changes the sign under  $\xi \rightarrow \gamma_5\xi$  and  $\psi(x) \rightarrow \gamma_5\psi(x)$ , but the latter does not.

then these produce combination (6.13) under  $r$ . These possibilities are excluded, however, because in our new total action  $S_{\text{tot}}$ ,  $M$ ,  $v_V(x)$  and  $v_A(x)$  appear always in the particular combination,  $M + v_V(x) + v_A(x)\gamma_5$ . In this way, we establishes relation (6.12).

In Eq. (6.12), the integer  $n \geq 1$  was the lowest-order of the loop expansion in which  $\mathcal{Z}_\chi^{O(\hbar^n)} \neq 0$  and  $\mathcal{Z}_P^{O(\hbar^n)} \neq 0$ . However, once Eq. (6.12) is fulfilled, then the consistency condition (6.11) applies to the next leading order  $O(\hbar^{n+1})$ . Thus, repeating the above argument, we have

$$\mathcal{Z}_\chi = \frac{1}{2}\mathcal{Z}_P, \quad (6.17)$$

to all orders of the loop expansion. This proves one of our assertions, Eq. (4.22).

### 6.5. $O(\theta^0, \xi^2)$ terms

Finally, we consider the  $O(\theta^0, \xi^2)$  terms in Eq. (6.3). This yields,

$$\begin{aligned} & \int d^4x \bar{\xi} [rX_S(x)^{O(\hbar^n)}]_{O(\theta^0, \xi^1)} \\ & + \int d^4x M \langle \bar{\psi}^a(x) [s^2\psi^a(x) - \gamma_5 \bar{\xi} \gamma_5 D\psi^a(x)] \rangle_{O(\theta^0, \xi^2)}^{O(\hbar^n)} \\ & + \int d^4x (-i) \bar{\xi} \gamma_\mu \xi \langle X_{t_\mu}(x) \rangle_{O(\theta^0, \xi^0)}^{O(\hbar^n)} = 0. \end{aligned} \quad (6.18)$$

Then, if we substitute Eqs. (4.17a)–(4.17e) in the first term, we have

$$\mathcal{Z}_{3F}^{O(\hbar^n)} = 0, \quad (6.19)$$

as the coefficient of the combination<sup>20</sup>

$$\frac{1}{4} \int d^4x \left\{ \bar{\xi} \sigma_{\mu\nu} \xi \text{tr} [F_{\mu\nu}(x) \bar{\psi}(x) \psi(x)] - \bar{\xi} \gamma_5 \sigma_{\mu\nu} \xi \text{tr} [F_{\mu\nu}(x) \bar{\psi}(x) \gamma_5 \psi(x)] \right\}. \quad (6.20)$$

In fact, it is easy to see without any calculation that among Eqs. (4.17a)–(4.17e) only the three-fermion operator (4.17d) contributes to Eq. (6.20):

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<sup>20</sup>We have noted that the coefficient  $\bar{\xi} \sigma_{\mu\nu} \xi$  is linearly independent of  $\bar{\xi} \gamma_\mu \xi$  (the coefficient of the last term of Eq. (6.18)) because under  $\xi \rightarrow \gamma_5 \xi$ , the former does not change the sign but the latter does.

Eq. (4.17d) is proportional to the totally-symmetric tensor (6.9), while others are proportional to  $\delta^{ab} = 2 \text{tr}(T^a T^b)$ .

Again, it is easy to see that the terms containing  $c(x)$ ,  $\bar{c}(x)$  or  $B(x)$ , Eq. (4.17f), do not contribute to Eq. (6.20).

Concerning the possible contribution of the BRS non-invariant terms (4.17g) to Eq. (6.20), the unique possibility is

$$\bar{\xi} \left[ r \frac{1}{2} J_{\mu\nu\rho\sigma} \gamma_\mu \text{tr} [\psi(x) \{A_\nu(x), F_{\rho\sigma}(x)\}] \right]_{O(\theta^0, \xi^1)}. \quad (6.21)$$

One sees that when  $J_{\mu\nu\rho\sigma} = d\delta_{\mu\rho}\delta_{\nu\sigma}$ , (the integral of) this is proportional to Eq. (6.20) by using the Fierz theorem (A.1). However, we have shown in Eq. (6.7) that Eq. (6.21) with  $J_{\mu\nu\rho\sigma} = d\delta_{\mu\rho}\delta_{\nu\sigma}$  cannot appear in the mixing of  $X_S(x)$  (4.17g). Thus, BRS non-invariant operators (4.17g) do not contribute to Eq. (6.20).

For the modification of Eqs. (4.17) in the presence of new external variables, what could contribute to Eq. (6.20) is the combination

$$\begin{aligned} X_S(x) \\ \sim \frac{u_V(x)}{M} \frac{1}{4} \{ \sigma_{\mu\nu} \xi \text{tr} [F_{\mu\nu}(x) \bar{\psi}(x) \psi(x)] - \gamma_5 \sigma_{\mu\nu} \xi \text{tr} [F_{\mu\nu}(x) \bar{\psi}(x) \gamma_5 \psi(x)] \}. \end{aligned} \quad (6.22)$$

This possibility is again excluded, because  $u_V(x)$  should appear only in the combination  $M + u_V(x)$ . In this way, we establish Eq. (6.19), and repeating the argument for higher  $n$ , we have

$$\mathcal{Z}_{3F} = 0. \quad (6.23)$$

That is, we establish Eq. (4.23).

## 7. Conclusion

In this paper, in the context of the lattice regularization of 4D  $\mathcal{N} = 1$  SYM, we formulated a generalized BRS transformation that treats the gauge, SUSY, translation and  $U(1)_A$  transformations in a unified way. On the basis of this (almost-nilpotent) transformation on lattice variables, we obtained a generalized WZ consistency condition for the symmetry breaking effects in the lattice formulation. Utilizing this powerful machinery, we then proved

that  $\mathcal{Z}_\chi = (1/2)\mathcal{Z}_P$  which implies the coincidence of the chiral symmetric limit (3.13) and the supersymmetric limit (4.21), and that  $\mathcal{Z}_{3F} = 0$  which implies that there is no exotic breaking of the SUSY WT identity by the three-fermion operator (4.19). Our these results provide a solid theoretical basis for lattice formulations of 4D  $\mathcal{N} = 1$  SYM. It is interesting to investigate further consequences of the consistency condition (5.39) or (6.3).

In the continuum theory, the generalized BRS symmetry has been formulated also for 4D  $\mathcal{N} = 4$  SYM [74] and for 4D  $\mathcal{N} = 2$  SYM [75, 76]; in the former, such a framework is crucially important because no off-shell multiplet is known. Adopting this framework to the lattice formulation might be useful to systematically classify the necessary fine-tuning in such formulations.<sup>21</sup>

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## Note added in the proof

The operator mixing coefficients  $\mathcal{Z}_P$  and  $\mathcal{Z}_\chi$  are dimensionless combinations of the bare gauge coupling constant  $g$  and the bare gluino mass  $M$ ,<sup>22</sup>

$$\mathcal{Z}_P = \mathcal{Z}_P(g, aM), \quad \mathcal{Z}_\chi = \mathcal{Z}_\chi(g, aM). \quad (7.1)$$

In the perturbation theory in the present paper, the mass  $M$  (not  $aM$ ) is treated as a fixed parameter and the combination  $aM$  is hence regarded

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<sup>21</sup>This problem has been studied for 4D  $\mathcal{N} = 4$  SYM in a recent paper [77].

<sup>22</sup>These power-divergence subtraction coefficients are independent of the renormalization point  $\mu$  [42].

as a higher-order quantity in the lattice spacing. What we have proven in Eq. (6.17) is thus the equality

$$\mathcal{Z}_\chi(g, 0) = \frac{1}{2}\mathcal{Z}_P(g, 0), \quad (7.2)$$

to all orders in the power series of  $g$ .

On the other hand, in actual numerical simulations with a small but fixed lattice spacing  $a$ , the chiral limit would be specified by the tuning,  $M \rightarrow M_{\text{cr}}^{\text{chiral}}(g)$ , where  $M_{\text{cr}}^{\text{chiral}}(g)$  is the solution of [41]

$$aM_{\text{cr}}^{\text{chiral}}(g) - \frac{1}{2}\mathcal{Z}_P(g, aM_{\text{cr}}^{\text{chiral}}(g)) = 0. \quad (7.3)$$

Similarly, the supersymmetric limit would be specified by,  $M \rightarrow M_{\text{cr}}^{\text{SUSY}}(g)$ , where [36]

$$aM_{\text{cr}}^{\text{SUSY}}(g) - \mathcal{Z}_\chi(g, aM_{\text{cr}}^{\text{SUSY}}(g)) = 0. \quad (7.4)$$

Then the question is whether from Eq. (7.2) one can draw any conclusion concerning the relation between  $M_{\text{cr}}^{\text{chiral}}(g)$  and  $M_{\text{cr}}^{\text{SUSY}}(g)$  which are defined by Eqs. (7.3) and (7.4), respectively. The answer is affirmative as follows: The point is that the mixing coefficients  $\mathcal{Z}_P$  and  $\mathcal{Z}_\chi$  are of  $O(g^2)$ , thus for a fixed  $aM$ ,  $\mathcal{Z}_P(g, aM)$  and  $\mathcal{Z}_\chi(g, aM)$  become arbitrarily small as  $g \rightarrow 0$ . This shows that, irrespective of how  $\mathcal{Z}_P$  and  $\mathcal{Z}_\chi$  depend on  $aM$ , we have

$$\lim_{g \rightarrow 0} aM_{\text{cr}}^{\text{chiral}}(g) = \lim_{g \rightarrow 0} aM_{\text{cr}}^{\text{SUSY}}(g) = 0. \quad (7.5)$$

In particular,  $aM_{\text{cr}}^{\text{chiral}}(g)$  and  $aM_{\text{cr}}^{\text{SUSY}}(g)$  vanish in the quantum continuum limit in which  $g \rightarrow 0$ . This allows us to approximate Eqs. (7.3) and (7.4) by

$$\begin{aligned} aM_{\text{cr}}^{\text{chiral}}(g) - \frac{1}{2}\mathcal{Z}_P(g, 0) &= 0, \\ aM_{\text{cr}}^{\text{SUSY}}(g) - \mathcal{Z}_\chi(g, 0) &= 0, \end{aligned} \quad (7.6)$$

as we approach to the continuum limit. Then Eq. (7.2) implies that  $M_{\text{cr}}^{\text{chiral}}(g)$  and  $M_{\text{cr}}^{\text{SUSY}}(g)$  coincide in the continuum limit.

## Appendix A. Three-fermion spinorial operators

In this appendix, we explore the relations between various gauge-invariant three-fermion operators and show that the operator  $\mathcal{O}_S$  (4.19) generally does not vanish.

For any  $4 \times 4$  matrices  $\Lambda^1$  and  $\Lambda^2$ , we have the Fierz theorem

$$\begin{aligned}\Lambda^1 \psi^a (\bar{\psi}^b \Lambda^2 \psi^c) &= \mp \frac{1}{4} \sum_{\lambda_A} \lambda_A \psi^c (\bar{\psi}^b \Lambda^2 \lambda_A \Lambda^1 \psi^a) \\ &= \frac{1}{4} \sum_{\lambda_A} \lambda_A \psi^c (\psi^{Ta} \Lambda^{1T} \lambda_A^T \Lambda^{2T} C^{-1} \psi^b),\end{aligned}\quad (\text{A.1})$$

where the upper sign holds when both  $\psi^a$  and  $\psi^b$  are Grassmann-odd and the lower holds otherwise;  $\lambda_A$  is the complete basis for  $4 \times 4$  matrices:

$$\lambda_A \equiv \{1, \gamma_5, \gamma_\alpha, i\gamma_5 \gamma_\alpha, i\sigma_{\alpha\beta}\}. \quad (\text{A.2})$$

In the last line of Eq. (A.1), we have used the constraint  $\bar{\psi}^b = \psi^{Tb}(-C^{-1})$  and  $C^T = -C$ . Since

$$\lambda_A^T \equiv c_A C^{-1} \lambda_A C, \quad (\text{A.3})$$

where

$$c_A = \{1, 1, -1, 1, -1\}, \quad (\text{A.4})$$

when  $\Lambda^1 = \Lambda^2 = \Lambda \in \lambda_A$ , we have

$$\Lambda \psi^a (\bar{\psi}^b \Lambda \psi^c) = -\frac{1}{4} \sum_A c_A \lambda_A \psi^c (\bar{\psi}^a \Lambda \lambda_A \Lambda \psi^b). \quad (\text{A.5})$$

Multiplying by  $\text{tr}(T^a T^b T^c)$  and summing over  $a, b$ , and  $c$ , we have

$$\Lambda \text{tr} [\psi (\bar{\psi} \Lambda \psi)] = -\frac{1}{4} \sum_A c_A \lambda_A \text{tr} [\psi (\bar{\psi} \Lambda \lambda_A \Lambda \psi)]. \quad (\text{A.6})$$

Applying this relation to

$$\begin{aligned}\mathcal{O}_S &\equiv \text{tr} [\psi (\bar{\psi} \psi)], \\ \mathcal{O}_P &\equiv \gamma_5 \text{tr} [\psi (\bar{\psi} \gamma_5 \psi)], \\ \mathcal{O}_V &\equiv \sum_{\mu} \gamma_{\mu} \text{tr} [\psi (\bar{\psi} \gamma_{\mu} \psi)], \\ \mathcal{O}_A &\equiv \sum_{\mu} i\gamma_5 \gamma_{\mu} \text{tr} [\psi (\bar{\psi} i\gamma_5 \gamma_{\mu} \psi)], \\ \mathcal{O}_T &\equiv \sum_{\mu < \nu} i\sigma_{\mu\nu} \text{tr} [\psi (\bar{\psi} i\sigma_{\mu\nu} \psi)],\end{aligned}\quad (\text{A.7})$$

and using

$$\begin{aligned}
1\lambda_A 1 &\equiv s_A \lambda_A, & s_A &= \{1, 1, 1, 1, 1\}, \\
\gamma_5 \lambda_A \gamma_5 &\equiv p_A \lambda_A, & p_A &= \{1, 1, -1, -1, 1\}, \\
\sum_{\mu} \gamma_{\mu} \lambda_A \gamma_{\mu} &\equiv v_A \lambda_A, & v_A &= \{4, -4, -2, 2, 0\}, \\
\sum_{\mu} i\gamma_5 \gamma_{\mu} \lambda_A i\gamma_5 \gamma_{\mu} &\equiv a_A \lambda_A, & a_A &= \{4, -4, 2, -2, 0\}, \\
\sum_{\mu < \nu} i\sigma_{\mu\nu} \lambda_A i\sigma_{\mu\nu} &\equiv t_A \lambda_A, & t_A &= \{6, 6, 0, 0, -2\},
\end{aligned} \tag{A.8}$$

we have

$$\begin{aligned}
\mathcal{O}_S &= -\frac{1}{4} \sum_A c_A s_A \lambda_A \operatorname{tr} [\psi (\bar{\psi} \lambda_A \psi)], \\
\mathcal{O}_P &= -\frac{1}{4} \sum_A c_A p_A \lambda_A \operatorname{tr} [\psi (\bar{\psi} \lambda_A \psi)], \\
\mathcal{O}_V &= -\frac{1}{4} \sum_A c_A v_A \lambda_A \operatorname{tr} [\psi (\bar{\psi} \lambda_A \psi)], \\
\mathcal{O}_A &= -\frac{1}{4} \sum_A c_A a_A \lambda_A \operatorname{tr} [\psi (\bar{\psi} \lambda_A \psi)], \\
\mathcal{O}_T &= -\frac{1}{4} \sum_A c_A t_A \lambda_A \operatorname{tr} [\psi (\bar{\psi} \lambda_A \psi)],
\end{aligned} \tag{A.9}$$

or, equivalently,

$$-4 \begin{pmatrix} \mathcal{O}_S \\ \mathcal{O}_P \\ \mathcal{O}_V \\ \mathcal{O}_A \\ \mathcal{O}_T \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & -1 & -1 \\ 4 & -4 & 2 & 2 & 0 \\ 4 & -4 & -2 & -2 & 0 \\ 6 & 6 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \mathcal{O}_S \\ \mathcal{O}_P \\ \mathcal{O}_V \\ \mathcal{O}_A \\ \mathcal{O}_T \end{pmatrix}. \tag{A.10}$$

By solving this (singular) simultaneous equation, we find

$$\mathcal{O}_V = \mathcal{O}_T = 0, \quad \mathcal{O}_P = -\mathcal{O}_S, \quad \mathcal{O}_A = -4\mathcal{O}_S, \tag{A.11}$$

and  $\mathcal{O}_S$  is undetermined.

With a help of a Mathematica package [78], we verified that the combination  $\mathcal{O}_S$  is in fact non-zero for the gauge group  $SU(3)$ , by explicitly expressing it in terms of Grassmann-odd variables  $\psi_\alpha^a$ . (When the gauge group is  $SU(2)$ , one immediately sees that  $\mathcal{O}_S = 0$  and hence  $\mathcal{O}_P = \mathcal{O}_A = 0$ .) Thus the combinations,  $\mathcal{O}_S$ ,  $\mathcal{O}_P$  and  $\mathcal{O}_A$ , can generally be non-vanishing.

## Appendix B. Triviality of dimension 9/2 $s_0$ -invariant operators that contain $c$ , $\bar{c}$ or $B$

In this appendix, we show the following<sup>23</sup>

**Lemma 1.** *Suppose  $\mathcal{O}(x)$  is an  $s_0$ -invariant operator with zero ghost-number that contains a gauge ghost  $c$ , an anti-ghost  $\bar{c}$  or an auxiliary field  $B$ . If its mass-dimension is 9/2 or less and if it behaves in the same way as  $\chi(x)$  (4.10) under lattice discrete symmetries,  $\mathcal{O}(x)$  is  $s_0$ -exact.*

*Proof.* To comply with the behavior under the hypercubic transformation,  $\mathcal{O}(x)$  must contain at least one  $\psi(x)$  whose mass-dimension is 3/2. From this and the fact that  $\mathcal{O}(x)$  is of zero ghost-number, it immediately follows that  $\mathcal{O}(x)$  is linear in  $\bar{c}(x)$  or in  $B(x)$ . Thus, the most general form of  $\mathcal{O}(x)$  is given by

$$\mathcal{O}(x) = \bar{c}^a(x)\Delta_1^a(x) - B^a(x)\Delta_0^a(x) + \partial_\mu \bar{c}^a(x)\Delta_{1\mu}^a(x) - \partial_\mu B^a(x)\Delta_{0\mu}^a(x), \quad (\text{B.1})$$

where  $\Delta$ 's are independent of  $\bar{c}$ . The  $s_0$ -invariance  $s_0\mathcal{O}(x) = 0$  of course implies  $s_0\mathcal{O}(x)|_{\bar{c}=0} = 0$  and, from this, we have  $\Delta_1^a(x) = s_0\Delta_0^a(x)$  and  $\Delta_{1\mu}^a(x) = s_0\Delta_{0\mu}^a(x)$ . Hence,

$$\mathcal{O}(x) = -s_0 [\bar{c}^a(x)\Delta_0^a(x) + \partial_\mu \bar{c}^a(x)\Delta_{0\mu}^a(x)]. \quad (\text{B.2})$$

□

## Appendix C. Useful identities

For a Grassmann-even spinor that obeys constraint (5.3), the following relations hold:

$$\bar{\xi}\xi = \bar{\xi}\gamma_5\xi = \bar{\xi}\gamma_5\gamma_\mu\xi = 0, \quad (\text{C.1})$$

---

<sup>23</sup>A corresponding statement (without a proof) can be found in Ref. [1] (just below Eq. (17)).



and

$$\begin{aligned}
\bar{\xi}\gamma_5\sigma_{\mu\nu}\xi &= -\epsilon_{\mu\nu\rho\sigma}\bar{\xi}\sigma_{\rho\sigma}\xi, \\
\bar{\xi}\gamma_\mu\gamma_\nu\gamma_\rho\xi &= \bar{\xi}(\delta_{\nu\rho}\gamma_\mu - \delta_{\mu\rho}\gamma_\nu + \delta_{\mu\nu}\gamma_\rho)\xi, \\
\bar{\xi}\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\xi &= 2\epsilon_{\mu\nu\rho\sigma}\bar{\xi}\gamma_\sigma\xi, \\
\bar{\xi}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\xi &= \bar{\xi}(\delta_{\mu\nu}\sigma_{\rho\sigma} - \delta_{\mu\rho}\sigma_{\nu\sigma} + \delta_{\mu\sigma}\sigma_{\nu\rho} + \delta_{\nu\rho}\sigma_{\mu\sigma} - \delta_{\nu\sigma}\sigma_{\mu\rho} + \delta_{\rho\sigma}\sigma_{\mu\nu})\xi.
\end{aligned}
\tag{C.2}$$

On the other hand, for a Grassmann-odd spinor that obeys constraint (2.7), we have

$$S^{ab}\bar{\psi}^a\gamma_\mu\psi^b = S^{ab}\bar{\psi}^a\sigma_{\mu\nu}\psi^b = S^{ab}\bar{\psi}^a\gamma_5\sigma_{\mu\nu}\psi^b = 0, \tag{C.3}$$

for a symmetric coefficient  $S^{ba} = S^{ab}$ , and

$$A^{ab}\bar{\psi}^a\psi^b = A^{ab}\bar{\psi}^a\gamma_5\psi^b = A^{ab}\bar{\psi}^a\gamma_5\gamma_\mu\psi^b = 0, \tag{C.4}$$

for an antisymmetric coefficient  $A^{ba} = -A^{ab}$ .

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